INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — Math Test #4 — 1h30

May 12, 2023

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

Exercise 1. Determine the domain of differentiability of the following functions as well as an expression of their derivatives:

$$f : \begin{array}{rcl} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & e^{\cos(x^2)} \end{array}. \qquad g : \begin{array}{rcl} D_g & \longrightarrow & \mathbb{R} \\ x & \longmapsto & (|x|-1)\arcsin(x) \end{array}$$

where D_g is the maximal subset of $\mathbb{R}$ that makes g well defined (that you will determine).

Exercise 2. Let

$$f : \begin{array}{rcl} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & x\arctan(x) - \frac{1}{2}\ln(1+x^2). \end{array}$$

You're given that f is well defined and of class C^∞.

1. Determine f', and deduce the variations of f.
2. a) Show that
$$\ln(1+x^2) \underset{x\to+\infty}{\sim} 2\ln x \underset{x\to+\infty}{=} o(x).$$
Hint. For $x > 0$, $\ln(1+x^2) = \ln\left(x^2\left(1+\frac{1}{x^2}\right)\right) = \cdots$

   b) Find the simplest equivalent as $x \to +\infty$ of $x\arctan(x)$.

   c) Deduce the simplest equivalent of f at $+\infty$, as well as the value of the limit $\lim\limits_{+\infty} f$.
3. We define
$$g : \begin{array}{rcl} \mathbb{R}_+ & \longrightarrow & [1,+\infty) \\ x & \longmapsto & f(x)+1. \end{array}$$

   a) Explain why g is well defined, and why g a bijection.

   b) Show that g^{-1} is differentiable on $(1,+\infty)$, and for $y \in (1,+\infty)$ determine an expression of $(g^{-1})'(y)$.
   Note. Since we don't have an expression for g^{-1}, *your answer for* $(g^{-1})'(y)$ *may be expressed in terms of* g^{-1}.

Exercise 3. *For this exercise, you're given that the limits* $\lim\limits_{+\infty} \sin$ *and* $\lim\limits_{+\infty} \cos$ *do not exist.*
Let $\beta \in \mathbb{R}_+^*$ and define:

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto \begin{cases} |x|^\beta \sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

1. For $x \in \mathbb{R}^*$ compute the value of $f'(x)$.
2. Show that if $\beta > 1$ then f is differentiable at 0, and determine the value of $f'(0)$.
3. Show that if $\beta \leq 1$ then f is not differentiable at 0.
4. Find the values of β for which f is of class C^1.
5. Give an example of a function that is differentiable on $\mathbb{R}$ but such that f is not of class C^1.

Exercise 4.

1. Fill in the blank with the appropriate Taylor–Young expansion (no justifications required):

$$e^x \underset{x\to 0}{=} \cdots + o(x^2) \qquad\qquad \sin(x) \underset{x\to 0}{=} \cdots + o(x^3)$$

2. From these, deduce a Taylor–Young expansion at 0 of the following function:

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto e^x \sin(x).$$

3. Deduce simple equivalents for the following expressions:

$$(1)\ f(x) \underset{x\to 0}{\sim} \qquad (2)\ f(x) - x \underset{x\to 0}{\sim} \qquad (3)\ f(x) - x - x^2 \underset{x\to 0}{\sim}$$

4. Deduce the value of the following limit:

$$\ell = \lim_{x\to 0} \frac{e^x \sin(x) - x - x^2}{x^3}.$$

Exercise 5.

1. Recall the Extreme Value Theorem.
2. Choose one of the hypotheses of the Extreme Value Theorem, and show that if it is not fulfilled then the theorem doesn't hold true anymore, by constructing a counter example (give an explicit function as well as its graph).