Exam n° 1 – 1 hour 30 minutes

Warm-up exercises (3 points)

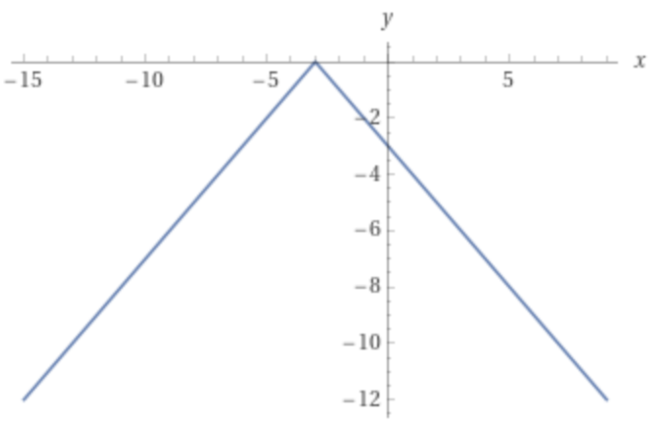
Exercise 1. Let $n \in \mathbb{N}$. Compute the following sum (justify the result): $\sum_{j=0}^{n} 2^j \binom{n}{j}$.

Solution.

- Using binomial theorem, we have $\forall a, b \in \mathbb{R}$, $(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j}$.
- In particular, for $a = 2$, $b = 1$ we obtain $3^n = \sum_{j=0}^{n} \binom{n}{j} 2^j 1^{n-j}$
- Then $\sum_{j=0}^{n} 2^j \binom{n}{j} = 3^n$.

□

Exercise 2. Sketch the graph of the function $f : x \mapsto -|x+3|$.



Solving equations (3 points)

Exercise 3. Solve in $\mathbb{R}$ the following inequality: $\sqrt{x} \leq 2 - x$.

Solution.

- Solving $\sqrt{x} \leq 2 - x$ implies $x \geq 0$ and $x \leq 2$ in order to be defined.
- Assume $x \in [0, 2]$. Then $\sqrt{x} \leq 2 - x \Longrightarrow x \leq 4 - 4x + x^2 \Longleftrightarrow x^2 - 5x + 4 \geq 0$.
- We solve $x^2 - 5x + 4 = 0$: $\Delta = 25 - 16 = 9$, then $x = \frac{5}{2} \pm \frac{3}{2}$.
- Therefore $x^2 - 5x + 4 \geq 0$ for $x \in (-\infty, 1] \cup [4, +\infty)$.
- Combining with earlier constraints we obtain solutions for $x \in [0, 1]$.

□

Exercise 4. Solve in $\mathbb{R}$ the following equation: $\cos\left(x+\frac{\pi}{4}\right)=\frac{\sqrt{3}}{2}$.

Solution.

- Using properties of known angles, we deduce that $x+\frac{\pi}{4}=\pm\frac{\pi}{6}+2\pi k$, $k\in\mathbb{Z}$
- We obtain $S=\{-\frac{5\pi}{12},-\frac{\pi}{12}\}+2\pi\mathbb{Z}$

□

Strategies of proof (8 points)

Exercise 5.

1. Show that $\forall n\in\mathbb{N}^*$, $-\frac{1}{n}+\frac{1}{(n+1)^2}+\frac{1}{n+1}\leq 0$.
2. Use previous question to show that $\forall n\in\mathbb{N}^*$, $\sum_{k=1}^{n}\frac{1}{k^2}\leq 2-\frac{1}{n}$.

Solution.

- Using a direct proof: given $n\geq 1$, $-\frac{1}{n}+\frac{1}{(n+1)^2}+\frac{1}{n+1}=-\frac{(n+1)^2}{n(n+1)^2}+\frac{n}{n(n+1)^2}+\frac{n(n+1)}{n(n+1)^2}=-\frac{1}{n(n+1)^2}\leq 0$ since $n(n+1)^2>0$.
- Let's prove by induction $\forall n\in\mathbb{N}^*$ $P(n)$ is true, where $P(n):=\sum_{k=1}^{n}\frac{1}{k^2}\leq 2-\frac{1}{n}$.

  For $n=1$: $\sum_{k=1}^{1}\frac{1}{k^2}=1\leq 2-1$. Then $P(1)$ is true.
  Given $n\geq 1$, assume $P(n)$ is true. Let us show that $P(n+1)$ is true.
  $\sum_{k=1}^{n+1}\frac{1}{k^2}=\sum_{k=1}^{n}\frac{1}{k^2}+\frac{1}{(n+1)^2}\leq 2-\frac{1}{n}+\frac{1}{(n+1)^2}$ (using $P(n)$).
  Using question 1 we have $-\frac{1}{n}+\frac{1}{(n+1)^2}\leq -\frac{1}{n+1}$, therefore $\sum_{k=1}^{n+1}\frac{1}{k^2}\leq 2-\frac{1}{n+1}$, which concludes the proof.

□

Exercise 6. The goal is to find all functions $f:\mathbb{R}\to\mathbb{R}$ so that:

$$\forall (x,y)\in\mathbb{R}^2, |f(x)+f(y)|=|x+y|. \qquad (E)$$

1. (a) We define $f_1:\left\{\begin{array}{ccc}\mathbb{R}&\to&\mathbb{R}\\ x&\mapsto&x\end{array}\right.$. Show that f_1 satisfies (E).

   (b) We define $f_2:\left\{\begin{array}{ccc}\mathbb{R}&\to&\mathbb{R}\\ x&\mapsto&-x\end{array}\right.$. Show that f_2 satisfies (E).
2. Let be $f:\mathbb{R}\to\mathbb{R}$.

   (a) Show that if $[(\forall x\in\mathbb{R}, f(x)=x)$ or $(\forall x\in\mathbb{R}, f(x)=-x)]$, then $\forall x\in\mathbb{R}, |f(x)|=|x|$.

(b) Write the negation of $[(\forall x \in \mathbb{R}, f(x) = x)$ or $(\forall x \in \mathbb{R}, f(x) = -x)]$.

(c) Show that if $(\forall x \in \mathbb{R}, |f(x)| = |x|)$, then $[\forall x \in \mathbb{R}, (f(x) = x$ or $f(x) = -x)]$ (read carefully).

3. Let be $f : \mathbb{R} \to \mathbb{R}$ a function satisfying (E).

   (a) Show that $f(0) = 0$.

   (b) Show that $\forall x \in \mathbb{R}, |f(x)| = |x|$.

   (c) Show by contradiction that we have $[(\forall x \in \mathbb{R}, f(x) = x)$ or $(\forall x \in \mathbb{R}, f(x) = -x)]$

4. What is the set of functions satisfying (E) then ?

Solution.

- $\forall x, y,\ |x + y| = |x + y|$ so f_1 satisfies (E).
- $\forall x, y,\ |-x + -y| = |x + y|$ so f_2 satisfies (E).
- If $\forall x \in \mathbb{R}, f(x) = x$ then $|f(x)| = x$, or if $\forall x \in \mathbb{R}, f(x) = -x$ then $|f(x)| = |-x| = |x|$. So if $\forall x \in \mathbb{R}, f(x) = x$ or $\forall x \in \mathbb{R}, f(x) = -x$ then $|f(x)| = |x|$.
- $\exists x \in \mathbb{R},\ f(x) \neq x$ and $\exists x' \in \mathbb{R},\ f(x') \neq -x'$.
- Assume $\forall x \in \mathbb{R}, |f(x)| = |x|$. Then

$$|f(x)| = \begin{cases} x & \text{if } x \geq 0, \\ x & \text{if } -x < 0 \end{cases} \iff \begin{cases} f(x) = x & \text{if } x \geq 0, f(x) \geq 0, \\ f(x) = -x & \text{if } x \geq 0, f(x) < 0, \\ f(x) = -x & \text{if } x < 0, f(x) \geq 0, \\ f(x) = x & \text{if } x < 0, f(x) < 0 \end{cases}$$

  then $\forall x \in \mathbb{R},\ f(x)$ takes only values x or $-x$, therefore $\forall x \in \mathbb{R},\ (f(x) = x$ or $f(x) = -x)$.

- Using (E) for $x = 0 = y$, $|f(0) + f(0)| = 0 \Longrightarrow f(0) = 0$.
- Using (E) for $x = y$, $|f(x) + f(x)| = |x + x| \iff |f(x)| = |x|$.
- Assume $[(\forall x_\in \mathbb{R}, f(x) = x)$ or $(\forall x \in \mathbb{R}, f(x) = -x)]$ is false. Using 2)b) $\exists x_0 \in \mathbb{R},\ f(x_0) \neq x_0$ and $\exists x_1 \in \mathbb{R},\ f(x_1) \neq -x_1$. Since $|f(x)| = |x|$ by 3)b), from question 2)c) we deduce that $\forall x \in \mathbb{R}, (f(x) = x$ or $f(x) = -x)$. Then $f(x_0) = -x_0$ and $f(x_!) = x_1$ and

$$\begin{aligned} |f(x_0) + f(x_1)| = |x_0 + x_1| &\iff |-x_0 + x_1| = |x_0 + x_1| \\ &\Longrightarrow \begin{cases} -x_0 + x_1 = x_0 + x_1 & \text{if } -x_0 + x_1 \geq 0, x_0 + x_1 \geq 0, \\ -x_0 + x_1 = -x_0 - x_1 & \text{if } -x_0 + x_1 \geq 0, x_0 + x_1 < 0, \\ x_0 - x_1 = x_0 + x_1 & \text{if } -x_0 + x_1 < 0, x_0 + x_1 \geq 0, \\ x_0 - x_1 = -x_0 - x_1 & \text{if } -x_0 + x_1 < 0, x_0 + x_1 < 0 \end{cases} \\ &\Longrightarrow \begin{cases} x_0 = 0 & \text{if } -x_0 + x_1 \geq 0, x_0 + x_1 \geq 0, \\ x_1 = 0 & \text{if } -x_0 + x_1 \geq 0, x_0 + x_1 < 0, \\ x_1 = 0 & \text{if } -x_0 + x_1 < 0, x_0 + x_1 \geq 0, \\ x_0 = 0 & \text{if } -x_0 + x_1 < 0, x_0 + x_1 < 0 \end{cases} \end{aligned}$$

  however $f(0) = \pm 0$ which contradicts the definition of x_0, x_1. Therefore $[(\forall x \in \mathbb{R}, f(x) = x)$ or $(\forall x \in \mathbb{R}, f(x) = -x)]$ is true.

- Using 3)c) we conclude that f satisfying $(E) \Longrightarrow \forall x \in \mathbb{R},\ f(x) = |x| \Longrightarrow f = f_1$ or $f = f_2$. The $S = \{f_1, f_2\}$.

□

Properties of functions (6 points)

Exercise 7. Let E be a non-empty set, and we denote two functions $f : E \to E$, $g : E \to E$. We define $h : E \to E$ so that: $\forall x \in E,\ h(x) = g(f(x) + x)$.

1. (a) Using quantifiers, write the proposition "h is injective", and "h is surjective".
   (b) If $\mathrm{Id}_E + f$ is injective and if g is injective, can we conclude that h is injective ? Justify your answer (we expect a clear reasoning).
2. (a) Let P, Q, R be three statements. Give the contrapositive of : $(P \vee Q) \Longrightarrow R$.
   (b) Consider the proposition

$$(P_0) \equiv [\ \mathrm{Id}_E + f \text{ is surjective or if } g \text{ is surjective}] \Longrightarrow h \text{ is surjective}$$

   Is (P_0) true ? Justify your answer (we expect a clear reasoning).

Solution.

- $\forall x, x' \in E,\ h(x) = h(x') \Longrightarrow x = x'$. In other words $\forall x, x' \in E,\ g(x + f(x)) = g(x' + f(x')) \Longrightarrow x = x'$
- $\forall y \in E, \exists x \in E,\ h(x) = y$. In other words $\forall y \in E, \exists x \in E,\ g(x + f(x)) = y$
- Let $x, x' \in E$ such that $g(x + f(x)) = g(x' + f(x'))$. Since g is injective, then $x + f(x) = x' + f(x') \Longleftrightarrow (\mathrm{Id}_E + f)(x) = (\mathrm{Id}_E + f)(x')$. Since $\mathrm{Id}_E + f$ is injective, we conclude that $x = x'$ therefore h is injective.
- $\neg R \Longrightarrow (\neg P \wedge \neg Q)$
- By case disjunction.
  If $\mathrm{Id}_E + f$ surjective but g is not surjective: $\forall z \in E, \exists x \in E,\ x + f(x) = z$ then $g(x + f(x)) = g(z)$. However $\exists y \in E, \forall z \in E,\ g(z) \neq y$, other words $g(x + f(x)) \neq y$. Then h is not surjective.

  If $\mathrm{Id}_E + f$ is not surjective but g is surjective: $\exists z' \in E, \forall x \in E,\ x + f(x) \neq z'$. Since $\forall x \in E, x + f(x) \in E$, by surjectivity of g, $\exists z \in E,\ g(z) = x + f(x)$ so $g(z) \neq z'$ which contradicts with g being surjective. So the statement [$\mathrm{Id}_E + f$ is not surjective but g is surjective] is false: either it's the case above, or both are not surjective. In all cases we conclude that h is not surjective. So (P_0) is not true.

□