Exam n° 1 – 1 hour 30 minutes

- No documents, no calculators, no cell phones or electronic devices allowed.
- Take a deep breath before starting (everything is going to be ok!) and read entirely the exam before starting. [1]
- All exercises are independent, you can do them in the order that you'd like.
- Please start an exercise at the top of a page (for readability).
- Number single pages, or simply the booklets (*copies doubles*) if multiple : for example 1/3, 2/3, 3/3
- All your answers must be fully (but concisely) justified, unless noted otherwise.
- Redaction and presentation matter! For instance, write full sentences and make sure your 'x' and 'n' can be distinguished.

Warm-up exercises (3 points)

Exercise 1. Let $n \in \mathbb{N}$. Compute the following sum (justify the result) : $\sum_{j=0}^{n} 2^j \binom{n}{j}$.

Exercise 2. Sketch the graph of the function $f : x \mapsto -|x+3|$.

Solving equations (3 points)

Exercise 3. Solve in $\mathbb{R}$ the following inequality : $\sqrt{x} \leq 2 - x$.

Exercise 4. Solve in $\mathbb{R}$ the following equation : $\cos\left(x + \frac{\pi}{4}\right) = \frac{\sqrt{3}}{2}$.

Strategies of proof (8 points)

Exercise 5.

1. Show that $\forall n \in \mathbb{N}^*, -\frac{1}{n} + \frac{1}{(n+1)^2} + \frac{1}{n+1} \leq 0$.
2. Show by induction that $\forall n \in \mathbb{N}^*, \sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n}$.

Exercise 6. The goal is to find all functions $f : \mathbb{R} \to \mathbb{R}$ so that :

$$\forall (x,y) \in \mathbb{R}^2, |f(x) + f(y)| = |x+y|. \qquad (E)$$

1. (a) We define $f_1 : \begin{cases} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x \end{cases}$. Show that f_1 satisfies (E).

1. Draw a star next to your name on the first page once this is done.

(b) We define $f_2 : \left\{ \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & -x \end{array} \right.$. Show that f_2 satisfies (E).

2. Let be $f : \mathbb{R} \to \mathbb{R}$.
   (a) Show that if $[(\forall x \in \mathbb{R}, f(x) = x)$ or $(\forall x \in \mathbb{R}, f(x) = -x)]$, then $\forall x \in \mathbb{R}, |f(x)| = |x|$.
   (b) Write the negation of $[(\forall x \in \mathbb{R}, f(x) = x)$ or $(\forall x \in \mathbb{R}, f(x) = -x)]$.
   (c) Show that if $(\forall x \in \mathbb{R}, |f(x)| = |x|)$, then $[\forall x \in \mathbb{R}, (f(x) = x$ or $f(x) = -x)]$ (read carefully).
3. Let be $f : \mathbb{R} \to \mathbb{R}$ a function satisfying (E).
   (a) Show that $f(0) = 0$.
   (b) Show that $\forall x \in \mathbb{R}, |f(x)| = |x|$.
   (c) Show by contradiction that we have $[(\forall x \in \mathbb{R}, f(x) = x)$ or $(\forall x \in \mathbb{R}, f(x) = -x)]$
4. What is the set of functions satisfying (E) then ?

Properties of functions (6 points)

Exercise 7.

Let E a non-empty set, and we denote two functions $f : E \to E$, $g : E \to E$. We define $h : E \to E$ so that : $\forall x \in E,\ h(x) = g(f(x) + x)$.

1. (a) Using quantifiers, write the proposition "h is injective", and "h is surjective".
   (b) If $\mathrm{Id}_E + f$ is injective and if g is injective, can we conclude that h is injective ? Justify your answer (we expect a clear reasoning).
2. (a) Let P, Q, R be three statements. Give the contrapositive of : $(P \vee Q) \Longrightarrow R$.
   (b) Consider the proposition

$$(P_0) \equiv [\ \mathrm{Id}_E + f \text{ is surjective or if } g \text{ is surjective}] \Longrightarrow h \text{ is surjective}$$

   Is (P_0) true ? Justify your answer (we expect a clear reasoning).