Exam n° 2023-2024 – 2 hours

- No documents, no calculators, no cell phones or electronic devices allowed.
- Take a deep breath before starting (everything is going to be ok!) and read entirely the exam before starting.[0].
- All exercises are independent, you can do them in the order that you'd like.
- Please start an exercise at the top of a page (for readability).
- Number single pages, or simply the booklets (*copies doubles*) if multiple : for example 1/3, 2/3, 3/3
- All your answers must be fully (but concisely) justified, unless noted otherwise.
- Redaction and presentation matter! For instance, write full sentences and make sure your 'x' and 'n' can be distinguished.
- Respecting all of the above is part of the exam grade.

Warm-up exercises (8 points)

You are expected to provide some steps for those exercises. Little partial credit will be given for just writing the answer.

Exercise 1. Consider the polynomial $P(X) = X^4 + 2X^3 - 2X - 1$. Factorize P in $\mathbb{R}$ and in $\mathbb{C}$.

Solution.
- We remark that 1 is root of P : $P(1) = 1 + 2 - 2 - 1 = 0$.
- We perform the euclidean division of P by $(X - 1)$:

$$
\begin{array}{r|l}
X^4 + 2X^3 - 2X - 1 & X - 1 \\ \cline{2-2}
-(X^4 - X^3) & X^3 + 3X^2 + 3X + 1 = (X+1)^3 \\ \cline{1-1}
3X^3 - 2X - 1 & \\
-(3X^3 - 3X^2) & \\ \cline{1-1}
3X^2 - 2X - 1 & \\
-(3X^2 - 3X) & \\ \cline{1-1}
X - 1 & \\
-(X - 1) & \\ \cline{1-1}
0 &
\end{array}
$$

- We find that $P(X) = (X - 1)(X + 3)^2$ (using the binomial theorem) both in $\mathbb{R}$ and $\mathbb{C}$.

□

Exercise 2. Provide the Taylor series expansion of order 2 for $f(x) = \cos(x) + \ln(1 - 4x)$ at $x = 0$.

Solution.
- The TSE of order 2 of $x \mapsto \cos(x)$ at $x = 0$ is $\cos(x) \underset{x\to 0}{=} 1 - \frac{x^2}{2} + h_2(x)x^2$, with $\lim\limits_{xto0} h_2(x) = 0$.
- The TSE of order 2 of $x \mapsto \ln(1 + x)$ at $x = 0$ is $\ln(1 + x) \underset{x\to 0}{=} x - \frac{x^2}{2} + g_2(x)x^2$, with $\lim\limits_{x\to 0} g_2(x) = 0$.

— By composition TSE of order 2 of $x \mapsto \ln(1-4x)$ at $x=0$ is $\ln(1-4x) \underset{x\to 0}{=} -4x-8x^2+g_2(x)x^2$, with $\lim_{x\to 0} g_2(x) = 0$ (abuse of notation here, we keep g_2).

— Therefore $f(x) \underset{x\to 0}{=} 1-4x-\dfrac{17x^2}{2}+f_2(x)x^2$, with $\lim_{x\to 0} f_2(x) = 0$.

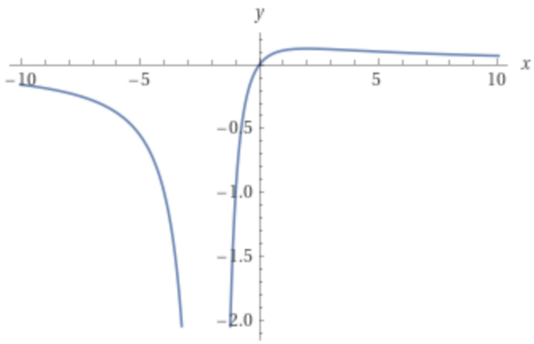
□

Exercise 3. Consider $f : A \to \mathbb{R}$ such that $f(x) = \dfrac{x}{x^2+4x+4}$. Provide the domain of definition A. Sketch the graph of f (justify limit behaviors). Is f surjective ? Is f injective ? Be as precise as possible in your answers.

Solution. — We remark that $f(x) = \dfrac{x}{(x+2)^2}$ so its domain of definition is $A = \mathbb{R}\setminus\{-2\}$.

— f is a rational fraction therefore $f(x) \sim_{\pm\infty} \dfrac{1}{x}$. Other other words $\lim_{x\to\pm\infty} f(x) = 0^{\pm}$.

— Additionally, $f(0) = 0$, for all $x \in \mathbb{R}_*^+$, $f(x) > 0$ and for all $x \in \mathbb{R}_*^- \setminus \{-2\}$, $f(x) < 0$. Finally



$\lim_{x\to -2^{\pm}} f(x) = -\infty$.

— f is not surjective : for example $f(x) = 1$ has no solution. Indeed : $f(x) = 1 \iff x^2+3x+4 = 0$, $\Delta = 9-16 < 0$.

— f is not injective either : for example $f(x) = -1 \iff x^2+5x+4 = 0$, $\Delta = 25-16 = 9$, $x = -\dfrac{5}{2} \pm \dfrac{3}{2}$, namely $x = -4$ or $x = -1$. We found $x \neq x'$ such that $f(x) = f(x')$.

□

Exercise 4. Consider $f : \mathbb{R}^4 \to \mathbb{R}^4$ such that

$$f(x,y,z,t) = (x+y+2z+t,\ x+2y+z-3t,\ 3x+y-z,\ x+12z+16t), \quad \forall (x,y,z,t) \in \mathbb{R}^4.$$

Find the preimage(s) by f of $(1,0,2,5)$, and give $\ker(f)$. We expect detailed steps and a proper solution written in the end.

Solution.

$$f\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 5 \end{pmatrix} \Leftrightarrow \begin{cases} x+y+2z+t = 1 \\ x+2y+z-3t = 0 \\ 3x+y-z = 2 \\ x+12z+16t = 5 \end{cases} \Leftrightarrow \begin{array}{c} \begin{array}{ccccc} x & y & z & t & b \end{array} \\ \left(\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & -3 & 0 \\ 3 & 1 & -1 & 0 & 2 \\ 1 & 0 & 12 & 16 & 5 \end{array}\right) \end{array}$$

$$L_2 \leftarrow L_2 - L_1$$
$$L_3 \leftarrow L_3 - 3L_1$$
$$L_4 \leftarrow L_4 - L_1$$

$$\overset{}{\Leftrightarrow}\begin{array}{c} \\ \\ \end{array}\left(\begin{array}{cccc|c} x & y & z & t & b \\ 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & -1 & -4 & -1 \\ 0 & -2 & -7 & -3 & -1 \\ 0 & -1 & 10 & 15 & 4 \end{array}\right) \begin{array}{l} L_3 \leftarrow L_3 + 2L_2 \\ L_4 \leftarrow L_4 + L_2 \\ \Leftrightarrow \end{array} \left(\begin{array}{cccc|c} x & y & z & t & b \\ 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & -1 & -4 & -1 \\ 0 & 0 & -9 & -11 & -3 \\ 0 & 0 & 9 & 11 & 3 \end{array}\right)$$

L_4 is the the opposite of L_3 we end up with a system of 3 equations for 4 unknowns. There is one free parameter.

$$\left(\begin{array}{cccc|c} x & y & z & t & b \\ 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & -1 & -4 & -1 \\ 0 & 0 & -9 & -11 & -3 \end{array}\right) \Leftrightarrow \begin{cases} x = 1 - \frac{4}{3}\lambda \\ y = -\frac{2}{3} + \frac{25}{9}\lambda \\ z = \frac{1}{3} - \frac{11}{9}\lambda \\ t = \lambda \end{cases}, \lambda \in \mathbb{R}$$

$$S = \left\{ \begin{pmatrix} 1 \\ -\frac{2}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -\frac{4}{3} \\ \frac{25}{9} \\ -\frac{11}{9} \\ 1 \end{pmatrix} : \lambda \in \mathbb{R} \right\} = \begin{pmatrix} 1 \\ -\frac{2}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix} + \mathrm{Span}\left(\begin{pmatrix} -\frac{4}{9} \\ \frac{25}{9} \\ -\frac{11}{9} \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 1 \\ -\frac{2}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix} + \ker(f)$$

□

Exercise 5. Consider $A = \{(x, y, z) \in \mathbb{R}^3 | x + y \geq 0,\ x + 2y + 3z = 0,\ x + y \leq 0\}$. Is A a vector subspace of $\mathbb{R}^3$? Justify your answer.

Solution. — First we remark that $\begin{cases} x + y \geq 0 \\ x + 2y + 3z = 0 \\ x + y \leq 0 \end{cases} \iff \begin{cases} x + y = 0 \\ x + 2y + 3z = 0 \end{cases}$

- $(0, 0, 0) \in A$
- Let $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3), \in A$, $\alpha, \beta \in \mathbb{R}$. Then $w := \alpha u + \beta v = (\alpha u_1 + \beta v_1, \alpha u_2 + \beta v_2, \alpha u_3 + \beta v_3) = (w_1, w_2, w_3)$. Let us prove that $w \in A$.
- $w_1 + w_2 = \alpha u_1 + \beta v_1 + \alpha u_2 + \beta v_2 = \alpha(u_1 + u_2) + \beta(v_1 + v_2) = 0 + 0$
- $w_1 + 2w_2 + w_3 = \alpha u_1 + \beta v_1 + 2\alpha u_2 + 2\beta v_2 + \alpha u_3 + \beta v_3 = \alpha(u_1 + 2u_2 + u_3) + \beta(v_1 + 2v_2 + v_3) = 0 + 0$
- Therefore A is a vector subspace
- Other option : $A = \{(x, y, z) \in \mathbb{R}^3 | x + y = 0,\ x + 2y + 3z = 0\} = \mathrm{Span}\left(\begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix}\right)$ so A is a vector subspace.

□

Functions (4 points)

Exercise 6.
For all $x \in \mathbb{R}$, define $f(x) = \mathrm{sh}\,(\sin(2x))$.

1. Provide a co-domain so that f is surjective.
2. Study the parity of f.
3. Is f injective ? Justify.
4. Without computing the derivative, show that f is monotonic on $[0, a]$, where $a > 0$ is to be determined.
5. Show that $\forall x \in [0, 1]$, $\mathrm{sh}\,(x) \geq x$.

6. Provide a restriction of f (denoted $\tilde{f}$) that is bijective.
7. Sketch the graph of $\tilde{f}$, as well as its reciprocal function.
8. (*For the challenge*) Find an expression of $\tilde{f}^{-1}$.

Solution.
— $f : \mathbb{R} \to [\mathrm{sh}\,(-1), \mathrm{sh}\,(1)]$ is surjective ($x \mapsto \mathrm{sh}\,(x)$ is continuous and increasing, $x \mapsto \sin(2x)$ is continuous and takes values in $[-1, 1]$).
— f is π-periodic : $\forall x \in \mathbb{R}$, $f(x + \pi) = \mathrm{sh}\,(\sin(2x + 2\pi)) = \mathrm{sh}\,(\sin(2x)) = f(x)$.
— f is not injective since it is periodic (we have found $x \neq x'$ such that $f(x) = f(x')$)
— By composition, $x \mapsto \sin(2x)$ is strictly increasing over $[0, \frac{\pi}{4}]$, and $x \mapsto \mathrm{sh}\,(x)$ is strictly increasing over $[0, \frac{\pi}{4}]$. So by composition f is strictly increasing (therefore monotonic) over $[0, a]$ with $a = \frac{\pi}{4}$.
— Define the function $g(x) = \mathrm{sh}\,(x) - x$. g is continuous and differentiable over $[0, 1]$ and $g'(x) = \mathrm{ch}\,(x) - 1 \geq 0$, $\forall x \; in[0, 1]$, Then we conclude that g is increasing and $\forall x \in [0, 1]$, $g(x) \geq g(0) = 0$. So $\mathrm{sh}\,(x) \geq x$.
— Since f is strictly monotonic over $[0, \frac{\pi}{4}]$, it is injective. Therefore $\tilde{f} : [0, \frac{\pi}{4}] \to [0, \mathrm{sh}\,(1)]$ is surjective and injective, it is a bijection.
— We know that $\tilde{f}$ is strictly increasing over $[0, \frac{\pi}{4}]$, and $\forall x \in [0, \frac{\pi}{4}]$, $X := \sin(2x) \in [0, 1]$ and $\tilde{f}(X) \geq X$.

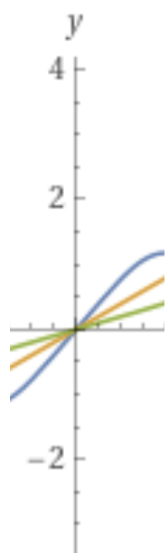


FIGURE 1 – Blue $\tilde{f}$, Orange $y = x$, Green $\tilde{f}^{-1}$.

— $\tilde{f}^{-1}(x) = \frac{1}{2} \arcsin(\mathrm{argsh}\,(x))$.

□

Polynomials (4 points)

Exercise 7.

Let $A = \{P \in \mathbb{R}_3[X] | \forall X, \; P(X + 1) - P(X) = X^2 - 1\}$.

1. Find all $P \in A$.

2. Without computing derivatives, provide $P^{(k)}(0)$, $k = 0, \dots, 3$.
3. Deduce $\sum_{k=0}^{n}(k-1)(k+1)$.
4. Given $P \in A$, show that there exists $a \in \mathbb{N}$ such that $P(a) = P(1+a)$ and $P(-a) = P(1-a)$.
5. Is the function $x \mapsto P(x)$ injective ? Justify your answer.

Solution. — $P \in \mathbb{R}_3[X]$ so we write $P(X) = aX^3 + bX^2 + cX + d$, $a, b, c, d \in \mathbb{R}$.

— Using the binomial theorem we find

$$\begin{aligned} P(X+1) - P(X) =& a(X^3 + 3X^2 + 3X + 1) + b(X^2 + 2X + 1) + c(X+1) + d \\ & - (aX^3 + bX^2 + cX + d) \\ =& 3aX^2 + (3a + 2b)X + (a + b + c) \\ =& X^2 - 1 \end{aligned}$$

by identification we find

$$\begin{cases} a = \frac{1}{3} \\ b = -\frac{3}{2}a = -\frac{1}{2} \\ c = -1 - a - b = -\frac{5}{6} \end{cases}$$

Therefore $A = \{P \in \mathbb{R}_3[X] |\, P(X) = \frac{1}{3}X^3 - \frac{1}{2}X^2 - \frac{5}{6}X + d,\ d \in \mathbb{R}\}$

— By Taylor we have $P(X) = P(0) + P'(0)X + \frac{P''(0)}{2}X^2 + \frac{P'''(0)}{6}X^3$, by identification we find

$$\begin{cases} P(0) = d \\ P'(0) = -\frac{5}{6} \\ P''(0) = -1 \\ P'''(0) = 2 \end{cases}$$

— $\forall X \in \mathbb{R}$, $P(X+1) - P(X) = X^2 - 1 = (X+1)(X-1)$. Therefore replace X by k and summing over $[\![0, n]\!]$ we get a telescopic sum :

$$\sum_{k=0}^{n}(k-1)(k+1) = \sum_{k=0}^{n} P(k+1) - \sum_{k=0}^{n} P(k) = P(n+1) - P(0) = \frac{1}{3}(n+1)^3 - \frac{1}{2}(n+1)^2 - \frac{5}{6}(n+1)$$

— We remark that $X^2 - 1$ has two roots : ± 1. Then $P(1+1) - P(1) = 0$ and $P(-1+1) - P(-1) = 0$, in other words $P(1) = P(1+1)$, and $P(-1) = P(-1+1)$ so $a = 1$.

— Using previous question $P(0) = P(1)$ so $x \mapsto P(x)$ is not injective (unless we restrict the domain).

□

Vector Subspace (4 points)

Exercise 8.

Consider the two subsets $F, G \subset \mathbb{R}^4$ given by :

$$F = \{(x, y, z, t) \in \mathbb{R}^4 |\, x + y + z = 0,\ x + y + t = 0\}, \quad G = \{(x, y, z, t) \in \mathbb{R}^4 |\, x = 0,\ y = 0\}.$$

0. Draw a cat next to your name on the first page once this is done.

1. Show that F, G are vector subspaces of $\mathbb{R}^4$ and determine a basis B_F of F, and a basis B_G of G.
2. Show that the family of vectors from B_F and B_G is a basis of $\mathbb{R}^4$. We will called this basis B'.
3. Given $u \in \mathbb{R}^4$ such that $[u]_B = (a, b, c, d)$ where B is the canonical basis. Give $[u]_{B'}$.

Solution. — $F = \text{Span}\left(\begin{pmatrix}1\\0\\-1\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\\-1\end{pmatrix}\right)$ and $G = \text{Span}\left(\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right)$ so F, G are vector subspaces of $\mathbb{R}^4$.

— $\left(\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right)$ is linearly independant (and generating F) so it is a basis of F. Same for $\left(\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right)$ so we conclude it is a basis of G.

— $B' = (B_F, B_G) = \left(\begin{pmatrix}1\\0\\-1\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right) = (u_1, u_2, e_3, e_4)$ where e_3, e_4 are canonical vectors.

$$\begin{aligned}
\text{rk}(B') &= \text{rk}\left(\begin{pmatrix}1\\0\\-1\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right)\\
&= \text{rk}\left(\begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-1\\-1\end{pmatrix}, \begin{pmatrix}1\\0\\-1\\-1\end{pmatrix}\right)\\
&= \text{rk}\left(\begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right) = \text{rk}(C) = 4
\end{aligned}$$

as we obtained a row echelon form ($B' \leftrightsquigarrow C$). Then B' is linearly independent and $\text{rk}(B') = 4 = \dim(\mathbb{R}^4)$ so it is a basis of $\mathbb{R}^4$.

— $[u]_B = ae_1 + be_2 + ce_3 + de_4$. We have

$$\begin{cases} u_1 = e_1 - e_3 - e_4 \\ u_2 = e_2 - e_3 - e_4 \\ u_3 = e_3 \\ u_4 = e_4 \end{cases} \iff \begin{cases} e_1 = -u_1 + u_3 + u_4 \\ e_2 = -u_2 + u_3 + u_4 \\ e_3 = u_3 \\ e_4 = u_4 \end{cases}$$

therefore $[u]_{B'} = -au_1 - bu_2 + (a+b+c)u_3 + (a+b+d)u_4 = \begin{pmatrix}-a\\-b\\a+b+c\\a+b+d\end{pmatrix}$

□