Exam n° 2023-2024 – 2 hours

- No documents, no calculators, no cell phones or electronic devices allowed.
- Take a deep breath before starting (everything is going to be ok!) and read entirely the exam before starting.[0].
- All exercises are independent, you can do them in the order that you'd like.
- Please start an exercise at the top of a page (for readability).
- Number single pages, or simply the booklets (*copies doubles*) if multiple : for example 1/3, 2/3, 3/3
- All your answers must be fully (but concisely) justified, unless noted otherwise.
- Redaction and presentation matter! For instance, write full sentences and make sure your 'x' and 'n' can be distinguished.
- Respecting all of the above is part of the exam grade. Provided rubric is indicative (changes may occur).

Warm-up exercises (8 points)

You are expected to provide some steps for those exercises. Little partial credit will be given for just writing the answer.

Exercise 1. Consider the polynomial $P(X) = X^4 + 2X^3 - 2X - 1$. Factorize P in $\mathbb{R}$ and in $\mathbb{C}$.

Exercise 2. Provide the Taylor series expansion of order 2 for $f(x) = \cos(x) + \ln(1-4x)$ at $x = 0$.

Exercise 3. Consider $f : A \to \mathbb{R}$ such that $f(x) = \dfrac{x}{x^2+4x+4}$. Provide the domain of definition A. Sketch the graph of f (justify limit behaviors). Is f surjective? Is f injective? Be as precise as possible in your answers.

Exercise 4. Consider $f : \mathbb{R}^4 \to \mathbb{R}^4$ such that

$$f(x,y,z,t) = (x+y+2z+t,\ x+2y+z-3t,\ 3x+y-z,\ x+12z+16t), \quad \forall (x,y,z,t) \in \mathbb{R}^4.$$

Find the preimage(s) by f of $(1,0,2,5)$, and give $\ker(f)$. We expect detailed steps and a proper solution written in the end.

Exercise 5. Consider $A = \{(x,y,z) \in \mathbb{R}^3 | x+y \geq 0,\ x+2y+3z = 0,\ x+y \leq 0\}$. Is A a vector subspace of $\mathbb{R}^3$? Justify your answer.

Functions (4 points)

Exercise 6.
For all $x \in \mathbb{R}$, define $f(x) = \operatorname{sh}(\sin(2x))$.

1. Provide a co-domain so that f is surjective.
2. Study the parity of f.
3. Is f injective? Justify.

4. Without computing the derivative, show that f is monotonic over $[0, a]$, where $a > 0$ is to be determined (provide the largest a possible).
5. Show that $\forall x \in [0, 1]$, $\text{sh}\,(x) \geq x$.
6. Provide a restriction of f (denoted $\tilde{f}$) that is bijective.
7. Sketch the graph of $\tilde{f}$, as well as its reciprocal function.
8. (*For the challenge*) Find an expression of $\tilde{f}^{-1}$.

Polynomials (4 points)

Exercise 7.

Let $A = \{P \in \mathbb{R}_3[X] | \forall X,\ P(X+1) - P(X) = X^2 - 1\}$.

1. Find all $P \in A$.
2. Without computing derivatives, provide $P^{(k)}(0)$, $k = 0, \ldots, 3$.
3. Deduce $\sum_{k=0}^{n} (k-1)(k+1)$.
4. Given $P \in A$, show that there exists $a \in \mathbb{N}^*$ such that $P(a) = P(1+a)$ and $P(-a) = P(1-a)$.
5. Is the function $x \mapsto P(x)$ injective ? Justify your answer.

Vector Subspace (4 points)

Exercise 8.
Consider the two subsets $F, G \subset \mathbb{R}^4$ given by :

$$F = \{(x, y, z, t) \in \mathbb{R}^4 |\, x + y + z = 0,\ x + y + t = 0\}, \quad G = \{(x, y, z, t) \in \mathbb{R}^4 |\, x = 0,\ y = 0\}.$$

1. Show that F, G are vector subspaces of $\mathbb{R}^4$ and determine a basis B_F of F, and a basis B_G of G.
2. Show that the family of vectors from B_F and B_G is a basis of $\mathbb{R}^4$. We will called this basis B'.
3. Given $u \in \mathbb{R}^4$, we write $[u]_B = (a, b, c, d)$ where B is the canonical basis. Give $[u]_{B'}$.

0. Draw a cat next to your name on the first page once this is done.