Exercise 1

1.	False. Take $f(x) = x^3$ on $(-1, 1)$. Then 0 is a critical point but not a local extremum.	
2.	True. By definition, since x a global maximum, then for all $t \in [a,b], f(t) \leq f(x)$. Now, taking any neighbourhood V of x, we have as a result that for all $t \in V, f(t) \leq f(x)$ since $V \subset [a,b]$, proving that x is a local minimum.	
3.	True. Since P is of class $\mathcal{C}^2$, then a is an inflection of P if and only if $P''(a) = 0$ and P'' changes sign around a. However, since P'' is of degree exactly 2, there are only 3 possibles cases : — P'' has no root and thus P has no inflection point — P'' has a single root, but keep the same sign on $\mathbb{R}$ so no inflection point for P either — P'' has two roots and change sign at each root, in which cases both roots are inflection points for P.	
4.	False. If the graph has a vertical tangent at $(a, f(a))$, then f is not differentiable at a	
5.	True. Let F be an antiderative of f. We know that $F(1)-F(0) = 0$. We set $g(x) = F(x+0.5)-F(x)$ which is continuous since F is continuous. We have $g(0) = F(0.5) - F(0)$ and $g(0.5) = F(1) - F(0.5) = F(0) - F(0.5) = -g(0)$. Using IVT, there exists $\alpha \in (0, 0.5)$ such that $g(0.5) = 0$, which is what we wanted to prove since $g(0.5) = \displaystyle\int_{\alpha}^{\alpha+0.5} f(t)dt$.	
6.	False. Take $f(x) = x$ on $[0, 1]$.	
7.	True. Assume by contradiction that for all $x \in [0,1]$, we have $f(x) \leq g(x)$. Then, by using property of integrals, we would have $\displaystyle\int_0^1 f(x)dx \leq \int_0^1 g(x)dx$, which contradicts our hypothesis. As a result, there must exists $c \in [0, 1]$ such that $f(c) > g(c)$.	
8.	False. Consider the function f defined on $[0, 2]$ as : $$f(x) = \begin{cases} x & \text{if } x \in [0,1) \\ x+1 & \text{if } x \in [1,2] \end{cases}$$ Then f is strictly increasing on $[0, 2]$ but is not bijective from $[0, 2]$ to $[f(0), f(3)] = [0, 3]$ since 1.5 has no pre-image. This statement becomes true if we add continuity as an hypothesis on f.	
9.	False. Let F be an antiderivative of f (that does exist since f is continous). We can rewrite g as $g(x) = F(x^2) - F(x)$. As such, g is differentiable and $g'(x) = 2xF'(x^2) - F(x) = 2xf(x^2) - f(x)$.	

Exercise 2

<table>
<tr><td>1.</td><td>We have $\lim\limits_{x\to 0^+} \mathrm{e}^{-1/x} = 0$, thus $\lim\limits_{0^+} f = 2$. Therefore f can be extended continously at 0.</td><td></td></tr>
<tr><td>2.</td><td>Let $x > 0$. We have $\dfrac{f(x)-f(0)}{x-0} = \dfrac{f(x)-2}{x} = \mathrm{e}^{-1/x} \underset{x\to 0^+}{\longrightarrow} 0$, hence f is differentiable at 0 and we have $f'(0) = 0$.</td><td></td></tr>
<tr><td>3.</td><td>We have $\dfrac{f(x)}{x} = \left(-\mathrm{e}^{-1/x} + \dfrac{2}{x}\right) \underset{x\to+\infty}{\sim} \longrightarrow -1$ thus $a = 1$.
Moreover, $f(x) + x = -x(\mathrm{e}^{-1/x} - 1) + 2$, and as $\mathrm{e}^t - 1 \underset{x\to+\infty}{\sim} t$, then $\mathrm{e}^{-1/x} - 1 \underset{x\to+\infty}{\sim} \dfrac{-1}{x}$.
So $-x(\mathrm{e}^{-1/x} - 1) \underset{x\to+\infty}{\sim} 1$. Consequently, $\lim\limits_{x\to+\infty} f(x) + x = 3$. Hence C_f has an asymptote near $+\infty$ with equation $y = -x + 3$.</td><td></td></tr>
<tr><td>4.</td><td>$\forall x > 0,\ f'(x) = -\mathrm{e}^{-1/x} - x \times \dfrac{1}{x^2}\mathrm{e}^{-1/x} = \left(-1 - \dfrac{1}{x}\right)\mathrm{e}^{-1/x}$.</td><td></td></tr>
<tr><td>5. (a)</td><td>The function f is continuous on $\mathbb{R}_+$, and since f' is < 0 on $\mathbb{R}_+^*$, f is strictly decreasing on $\mathbb{R}_+$. Therefore, it establishes a bijection from $\mathbb{R}_+$ to $J = f(\mathbb{R}_+) = (\lim\limits_{+\infty} f, f(0)] = (-\infty, 2]$.</td><td></td></tr>
<tr><td>(b)</td><td><table><tr><td>x</td><td>$-\infty$ 2</td></tr><tr><td>$f^{-1}(x)$</td><td>$+\infty$ ↘ 0</td></tr></table></td><td></td></tr>
<tr><td>(c)</td><td>We know that $C_{f^{-1}}$ is the image of C_f under the symmetry with respect to the line $y = x$. Now, the line with the equation $y = -x + 3$ is an asymptote of C_f at $+\infty$, and this line is perpendicular to the line with the equation $y = x$. Therefore, the line with the equation $y = -x + 3$ is an asymptote of $C_{f^{-1}}$ at $-\infty$.</td><td></td></tr>
<tr><td>(d)</td><td>We deduce from previous question that $f^{-1}(x) \underset{x\to-\infty}{=} -x + o(x)$, and therefore $f^{-1}(x) \underset{x\to-\infty}{\sim} -x$.</td><td></td></tr>
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Exercise 3

1.	$I_0 = \int_0^1 \frac{x}{2}\mathrm{d}x = \frac{1}{4}$ and $I_2 = \int_0^1 \frac{x}{1+x^2}\mathrm{d}x = \left[\frac{1}{2}\ln(1+x^2)\right]_0^1 = \frac{1}{2}\ln 2.$	
2. (a)	For all $x \in [0,1]$, $f_n(x) \geq 0$ and the integral bounds are in increasing order hence $I_n \geq 0$.	
(b)	For all $n \in \mathbb{N}$, $I_{n+1} - I_n = \int_0^1 (f_{n+1}(x) - f_n(x))\mathrm{d}x = \int_0^1 \frac{(1-x)x^{n+1}}{(1+x^n)(1+x^{n+1})}\mathrm{d}x \geq 0$ since we are taking the integral of a positive function on $[0,1]$. As a result, the sequence (I_n) is increasing.	
3. (a)	For all $n \in \mathbb{N}$, $J_n = \int_0^1 \frac{x^{n+1}}{1+x^n}\,dx$. Now, for all $x \in [0,1]$, $0 \leq \frac{x^{n+1}}{1+x^n} \leq x^{n+1}$. We deduce : $0 \leq J_n \leq \int_0^1 x^{n+1}\,dx$ that is $0 \leq J_n \leq \frac{1}{n+2}$. By the squeeze theorem, we have $\lim_{n\to+\infty} J_n = 0$.	
(b)	For all $n \in \mathbb{N}$, $J_n = \int_0^1 x\,\mathrm{d}x - I_n$ hence $I_n = \frac{1}{2} - J_n$. Therefore we have $\lim_{n\to+\infty} I_n = \frac{1}{2}$.	
4. (a)	For all $n \in \mathbb{N}^*$, $\int_0^1 \frac{x^{n+1}}{1+x^n}\,\mathrm{d}x = \int_0^1 x^2 \times \frac{x^{n-1}}{1+x^n}\,\mathrm{d}x = \int_0^1 u(x) \times v'(x)\,\mathrm{d}x$ avec $u(x) = x^2$ et $v(x) = \frac{1}{n}\ln(1+x^n)$. Since the functions u and v are C^1 on $[0,1]$, we can apply the integration by parts formula, which gives : $\int_0^1 \frac{x^{n+1}}{1+x^n}\,\mathrm{d}x = \left[\frac{x^2}{n}\ln(1+x^n)\right]_0^1 - \int_0^1 \frac{2x}{n} \times \ln(1+x^n)\mathrm{d}x = \frac{\ln 2}{n} - \frac{2}{n}\int_0^1 x\ln(1+x^n)\mathrm{d}x.$	
(b)	We can rewrite the result of the last question as : $\forall n \in \mathbb{N}^*$, $J_n = \frac{\ln 2}{n} - \frac{2K_n}{n}$. Since $\lim_{n\to+\infty} K_n = 0$, we have $\frac{2K_n}{n} \underset{n\to+\infty}{=} o\left(\frac{\ln 2}{n}\right)$ and thus $J_n \underset{n\to+\infty}{\sim} \frac{\ln 2}{n}$. Eventually, $I_n - \frac{1}{2} \underset{n\to+\infty}{\sim} -\frac{\ln 2}{n}$.	