INSTITUT NATIONAL DES SCIENCES APPLIQUEES LYON

FIMI, SCAN 1A *22nd of March 2024*

Mathematics Written Examination # 4

Duration: 1 hour 30 minutes. *All documents and electronic devices are prohibited.*

— *No documents, no calculators, no cell phones or electronic devices allowed.*
— *Take a deep breath before starting (everything is going to be ok !) and read entirely the exam before starting.*[0]*.*
— *All exercises are independent, you can do them in the order that you'd like.*
— *Please start an exercise at the top of a page (for readability).*
— *Number single pages, or simply the booklets (copies doubles) if multiple: for example 1/3, 2/3, 3/3*
— *All your answers must be fully (but concisely) justified, unless noted otherwise.*
— *Redaction and presentation matter ! For instance, write full sentences and make sure your 'x' and 'n' can be distinguished.*
— ***Respecting all of the above is part of the exam grade (0.75 points).*** *Provided rubric is indicative (changes may occur).*

Exercise 1 ($\sim$ 7 points) True or false

For all the following propositions, determine whether the proposition is true or false. You are expected to provide some justification for those questions (drawing might be used as part of the justification). Little to no credit will be given for just writing the answer True/False.

1. Let f be a differentiable function on (a, b) and let $x \in (a, b)$ be a critical point of f. Then x is a local extremum of f.

2. Let f be a differentiable function on $[a, b]$. Let x be a point where the global maximum of f is reached. Then x is a local maximum of f.

3. A polynomial $P \in \mathbb{R}[X]$ of degree four has either two inflection points or none.

4. A function f defined in a neighborhood of a point $a \in \mathbb{R}$ is differentiable at a if and only if the graph of f has a tangent at $(a, f(a))$.

5. For any continous function f defined on $[0, 1]$ such that $\int_0^1 f(t)dt = 0$, there exists $a \in [0, 0.5]$ such that $\int_a^{a+0.5} f(x)dx = 0$.

6. Let f be a differentiable function on $[a, b]$. Let x be a point where the global maximum of f is reached. Then x is a critical point of f.

7. For any f and g two continuous functions such that $\int_0^1 f(x)dx > \int_0^1 g(x)dx$, there exists $a \in [0, 1]$ such that $f(a) > g(a)$.

8. Any strictly increasing real function on $[a, b]$ is bijective from $[a, b]$ to $[f(a), f(b)]$.

9. Let f be a continuous function defined on $\mathbb{R}$ and let $g(x) = \int_x^{x^2} f(t)dt$. Then F is differentiable on $\mathbb{R}$ and $g'(x) = f(x^2) - f(x)$.

Exercise 2 ($\sim$ 7 points)

Let f be defined on $]0,+\infty[$ by : $\forall x>0,\ f(x)=-x\mathrm{e}^{-1/x}+2$.

1. Show that f can be extended continuously at 0.
   In the rest of the exercise, we still denote by f the function thus extended.
2. Study the differentiability of f at 0.
3. Show that the curve C_f of f has an oblique asymptote Δ at $+\infty$, whose equation will be determined.
4. For $x>0$, calculate $f'(x)$.
5. (a) Show that f realizes a bijection from $I=\mathbb{R}_+$ to an interval J to be determined.
   (b) Without justification, provide the complete variation table of f^{-1}.
   (c) Justify **using a graphical argument** that the curve $C_{f^{-1}}$ of f^{-1} has an oblique asymptote in the neighborhood of $-\infty$, for which you will provide an equation.
   (d) Deduce an equivalent of f^{-1} at $-\infty$ of the form αx^{β} with α and β non-zero real numbers.

Exercise 3 ($\sim$ 6 points) For every $n\in\mathbb{N}$, we define the function $f_n : x\mapsto \dfrac{x}{1+x^n}$ and the integral $I_n=\displaystyle\int_0^1 \frac{x}{1+x^n}\,dx$.

1. Calculate I_0 and I_2.
2. (a) Let $n\in\mathbb{N}$. Determine the sign of I_n.
   (b) Determine the direction of variation of the sequence $(I_n)_{n\in\mathbb{N}}$.
3. For every $n\in\mathbb{N}$, let $J_n=\displaystyle\int_0^1 \big(x-f_n(x)\big)\,dx$.
   (a) Using bounding, show that $\displaystyle\lim_{n\to+\infty} J_n=0$.
   (b) Hence, deduce the limit of I_n as n tends to infinity.
4. (a) Show that for all $n\in\mathbb{N}^*$,
$$\int_0^1 \frac{x^{n+1}}{1+x^n}\,dx=\frac{\ln 2}{n}-\frac{2}{n}\int_0^1 x\ln(1+x^n)\,dx.$$
   (b) For every $n\in\mathbb{N}$, let $K_n=\displaystyle\int_0^1 x\ln(1+x^n)\,dx$.
   It is admitted that $\displaystyle\lim_{n\to+\infty} K_n=0$.
   Hence deduce an equivalent of $I_n-\dfrac{1}{2}$ as n tends to $+\infty$.

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0. Draw a Minion next to your name on the first page once this is done.