Exam n° 1 – 1 hour 30 minutes

- No documents, no calculators, no cell phones or electronic devices allowed.
- Take a deep breath before starting (everything is going to be ok!) and read entirely the exam before starting. [1]
- All exercises are independent, you can do them in the order that you'd like.
- Please start an exercise at the top of a page (for readability).
- Number single pages, or simply the booklets (*copies doubles*) if multiple : for example 1/3, 2/3, 3/3
- All your answers must be fully (but concisely) justified, unless noted otherwise.
- Redaction and presentation matter! For instance, write full sentences and make sure your 'x' and 'n' can be distinguished.
- Questions marked (*) are considered more challenging.
- **Respecting all of the above is part of the exam grade (0.5 points).** Provided rubric is indicative (changes may occur).
- This exam is out 18.5 points. The additional 1.5 points come from your URCHIN quiz grade.

Warm-up exercises (9 points)

You are expected to provide some steps for those exercises. Little partial credit will be given for just writing the answer.

Exercise 1. True or False Justify briefly why the statement is True or False. If the statement is false, provide the correct answer.

1. The negation of the implication $B \Rightarrow A$ is $\neg A \Rightarrow B$.
2. The negation of $(\forall x \in E, \exists y \in E | f(x) + f(y) \geq 2)$ is $(\exists x \in E, \exists y \in E | f(x) + f(y) < 2)$.
3. The set of solutions of the inequation $\sqrt{2x+3} \geq x$ is $[-1, 3]$.
4. Consider the sets A,B,E with $A, B \subset E$. Then $E \setminus (A \cup B) = (E \setminus A) \cup (E \setminus B)$.
5. For all $a > 0$, $x \mapsto a^x$ and $x \mapsto x^a$ have the same monotonicity over $\mathbb{R}_*^+$.

Exercise 2.
Let $n \in \mathbb{N}^*$.

1. Show that for all $k = 1, 2, \ldots, n$, $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$
2. Deduce that $\sum_{k=0}^{n} k \binom{n}{k} = n2^{n-1}$.
3. Show by induction that $\forall n \geq 1$, $\sum_{k=0}^{n} k^3 = \left(\sum_{k=0}^{n} k\right)^2$ (Hint : recall the value of $\sum_{k=0}^{n} k$).

Exercise 3.

1. Solve the equation $\sin(3x) = \sin(2x)$ in $\mathbb{R}$.
2. Using addition and duplication formula, show that $\sin(3x) = \sin(x)\left(4\cos^2(x) - 1\right)$.
3. (*) Using previous questions, find the value of $\cos(\frac{\pi}{5})$.

1. Draw a jack-o'-lantern next to your name on the first page once this is done.

Logic (3.5 points)

Exercise 4.

Let $n \in \mathbb{N}^*$, and let $(n+1)$ real numbers $x_0, x_1, \ldots, x_n \in [0,1]$ such that $0 \leq x_0 \leq x_1 \leq \cdots \leq x_n \leq 1$. We want to show that the following proposition

$P \equiv$There exists at least 2 successive numbers out of these real numbers such that their distance is less than $\dfrac{1}{n}$

is true.

1. Rewrite P using mathematical language, with quantifiers and values $x_i - x_{i-1}$ for instance.
2. Write the negation of P.
3. Show P using a proof by contradiction. Provide steps.

Functions (5,5 points)

Exercise 5.

We consider the function $f : E \to F$ defined by $f(x) = \cosh(\ln(x))$.

1. Provide the domain of definition E of f, and the co-domain F corresponding to the direct image of the domain.
2. For $x \in E$, simplify the expression of f.
3. Can the function be extended by continuity at 0 ? Justify your answer.
4. **Injectivity :** Consider $f_1 : E_1 \to F_1$, $f_2 : E_2 \to E_1$ two injective functions.
   (a) Recall the definition of f_1 being injective.
   (b) Show that $f_1 \circ f_2$ is injective. Provide steps.
   (c) Assume f_1, f_2 are monotonic (over their respective domains). What can you say about the monotonicity of $f_1 \circ f_2$?
   (d) Provide a restriction $\tilde{f}$ of f so that it is injective. Justify your answer (if multiple choices, select the least restricted option).
5. Is $\tilde{f}$ surjective ? What do you conclude ?
6. What can you say about the min, max, inf, sup of $\tilde{f}$ over its domain ?
7. (*) Provide an expression of the inverse (if defined).