INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — S1 — Math Test #2 — 1h30 *December 2, 2016*

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

Exercise 1. Determine the maximal subset A of $\mathbb{R}$ such that

$$\forall x \in A,\ f(x) = \ln\left(\frac{1+\tanh(x)}{1-\tanh(x)}\right) \text{ is well-defined.}$$

Let $x \in A$. Simplify as much as possible the expression $f(x)$.

Exercise 2.

1. For a polynomial function $P : \mathbb{C} \to \mathbb{C}$ we define the proposition:

   $$(*) \qquad \forall z \in \mathbb{C},\ P(z+1) = P(z).$$

   The goal of this question is to determine all the polynomial functions P that satisfy Proposition $(*)$.

   a) Find a *simple* example of a polynomial function $P : \mathbb{C} \to \mathbb{C}$ that *doesn't* satisfy Proposition $(*)$.

   b) Check that if $P : \mathbb{C} \to \mathbb{C}$ is a *constant* polynomial function, then P satisfies Proposition $(*)$.

   c) Let P be a polynomial function that satisfies Proposition $(*)$.

      i) Assume that $z_0 \in \mathbb{C}$ is a root of P. Show that $z_0 + 1$ is also a root of P.

      ii) Deduce that if P has a root in $\mathbb{C}$ then P is the nil polynomial function.

   d) Deduce all the polynomial functions $P : \mathbb{C} \to \mathbb{C}$ that satisfy Proposition $(*)$.

2. For a polynomial function $Q : \mathbb{C} \to \mathbb{C}$ we define the proposition:

   $$(**) \qquad \forall z \in \mathbb{C},\ (z-2)Q(z+1) = (z+1)Q(z).$$

   The goal of this question is to determine all the polynomial functions Q that satisfy Proposition $(**)$.

   a) Let $Q : \mathbb{C} \to \mathbb{C}$ be a polynomial function that satisfies Proposition $(**)$.

      i) Show that 2 and 0 are roots of Q, and deduce that 1 is also a root of Q.

      ii) Explain why there exists a polynomial function $P : \mathbb{C} \to \mathbb{C}$ such that

      $$\forall z \in \mathbb{C},\ Q(z) = z(z-1)(z-2)P(z).$$

      iii) Using the fact that Q satisfies Proposition $(**)$, show that P satisfies Proposition $(*)$.

   b) Conversely, let $P : \mathbb{C} \to \mathbb{C}$ be a polynomial function that satisfies Proposition $(*)$, and define the polynomial function[1]

   $$Q : \begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{C} \\ z & \longmapsto & z(z-1)(z-2)P(z). \end{array}$$

   Show that Q satisfies Proposition $(**)$.

   c) Deduce all the polynomial functions $Q : \mathbb{C} \to \mathbb{C}$ that satisfy Proposition $(**)$.

[1] You don't have to justify that Q is a polynomial function.

Exercise 3. We define the function

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto \frac{x^2+2}{3}.$$

Let $u_0 \in (0,1)$ and define the sequence $(u_n)_{n\in\mathbb{N}}$ by

$$\forall n \in \mathbb{N},\ u_{n+1} = f(u_n).$$

The goal of this exercise is to show that 1 is the least upper bound of the sequence $(u_n)_{n\in\mathbb{N}}$.

1. Prove that the function f is increasing on $\mathbb{R}_+$.
2. Show that $f\big((0,1)\big) \subset (0,1)$.
3. Deduce that
$$\forall n \in \mathbb{N},\ u_n \in (0,1).$$
4. Deduce that 1 is an upper bound of the sequence $(u_n)_{n\in\mathbb{N}}$.
5. Show that
$$\forall n \in \mathbb{N},\ u_{n+1} - u_n = \frac{(1-u_n)(2-u_n)}{3}.$$
6. Deduce that the sequence $(u_n)_{n\in\mathbb{N}}$ is increasing.
7. Show that
$$\forall n \in \mathbb{N}^*,\ u_n = u_0 + \sum_{k=0}^{n-1}(u_{k+1} - u_k).$$
8. Let $M \in \mathbb{R}$ be the least upper bound of the sequence $(u_n)_{n\in\mathbb{N}}$.
   a) Explain why $M \leq 1$.
   b) Show that
   $$\forall k \in \mathbb{N},\ u_{k+1} - u_k \geq \frac{(1-M)(2-M)}{3}.$$
   c) Deduce that
   $$\forall n \in \mathbb{N}^*,\ u_n \geq n\frac{(1-M)(2-M)}{3}.$$
   d) Show that the proposition "$M < 1$" is false, and deduce that 1 is the least upper bound of the sequence $(u_n)_{n\in\mathbb{N}}$.

Exercise 4.

1. Determine the general real solution of the following differential equation:
$$3f' + f = 4.$$
2. a) Determine the general real solution of the following differential equation:
   $$f'' + 4f' + 13f = 0.$$
   b) Determine a particular solution of the following differential equation:
   $$f'' + 4f' + 13f = 1.$$
   c) Determine the general real solution of the following differential equation:
   $$f'' + 4f' + 13f = 1.$$
   d) Determine the solution(s) of the following IVP:
   $$\begin{cases} f'' + 4f' + 13f = 1 \\ f(0) = 1 \\ f'(0) = 0. \end{cases}$$

Exercise 5. Show that $\sup\big((-\infty, 1)\big) = 1$.