SCAN 1 — S1 — Math Test #3 — 1h30 *January 6, 2017*

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

Exercise 1.

1. Determine, if they exist, the value of following limits:

$$(1)\ \lim_{x\to\pi/6}\frac{2\sin(x)-1}{6x-\pi}, \qquad (2)\ \lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}.$$

2. Let $(u_n)_{n\in\mathbb{N}}$ be the sequence defined by

$$\forall n\in\mathbb{N},\ u_n=\frac{n-2}{n+1}.$$

Determine, if they exist, the value of the following limits:

$$(3)\ \lim_{n\to+\infty}u_n, \qquad (4)\ \lim_{n\to+\infty}u_n^n.$$

Exercise 2. We define the function f by:

$$\begin{array}{rcl} f : \mathbb{R}^* & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \dfrac{\sin^2(x)}{x\big(1+|x|\big)}. \end{array}$$

1. Show that f can be extended by continuity at 0. We denote by $\tilde{f}:\mathbb{R}\to\mathbb{R}$ this extension by continuity.
2. Show that $\tilde{f}$ is differentiable at 0 and determine the value of $\tilde{f}'(0)$. Deduce an equation of the tangent line to the graph of $\tilde{f}$ at $\big(0,\tilde{f}(0)\big)$.

Exercise 3.

1. Find all the elements $x\in\mathbb{R}$ such that the following expression

$$\frac{\arcsin(x)}{\arccos(x)}$$

is well-defined. We denote by D the set of all such elements x, and by f the corresponding function:

$$\begin{array}{rcl} f : D & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \dfrac{\arcsin(x)}{\arccos(x)}. \end{array}$$

2. For $x\in D$, give an expression of $f(x)$ that depends only on $\arccos(x)$. We recall that

$$\arcsin+\arccos=\frac{\pi}{2},$$

and you may use this fact without any justifications.

3. Deduce the variations of f.
4. Is f continuous? justify (briefly!) your answer.
5. Use a theorem covered in Chapter *Continuity* to determine the range $J=f(D)$ of f.
6. We denote by g the function f with the codomain altered to be its range:

$$\begin{array}{rcl} f : D & \longrightarrow & J \\ x & \longmapsto & f(x). \end{array}$$

Explain why g is a bijection.

7. Determine g^{-1} explicitly.

Exercise 4. The goal of this exercise is to show that the limit $\lim_{n\to+\infty} \cos(n)$ doesn't exist.
Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be the sequences defined by

$$\forall n \in \mathbb{N},\ u_n = \cos(n), \qquad v_n = \sin(n).$$

We proceed by contradiction and assume that the limit $\lim_{n\to+\infty} u_n$ exists in $\overline{\mathbb{R}}$, say $\lim_{n\to+\infty} u_n = \ell \in \overline{\mathbb{R}}$.

1. Explain why $\ell \in [-1, 1]$.
2. Explain why $\ell = \lim_{n\to+\infty} u_{2n}$.
3. Use the double angle formula to show that
$$2\ell^2 - \ell - 1 = 0.$$
4. Deduce the possible values for ℓ.
5. Let $n \in \mathbb{N}$. Express u_{n+1} in terms of u_n and v_n, and deduce that the sequence $(v_n)_{n\in\mathbb{N}}$ converges to a value ℓ' that you will explicit in terms of ℓ.
6. Use the Pythagorean theorem to obtain another relation between ℓ and ℓ'.
7. Find a contradiction and conclude.

Exercise 5 (*Arithmetico–Geometric Mean*)**.** Let $a, b \in \mathbb{R}$ such that $0 < a < b$. We define the sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ by

$$u_0 = a,\ v_0 = b, \qquad \forall n \in \mathbb{N},\ \begin{cases} u_{n+1} = \sqrt{u_n v_n} \\ v_{n+1} = \dfrac{u_n + v_n}{2}. \end{cases}$$

1. Show that for all $n \in \mathbb{N}$, $u_n > 0$ and $v_n > 0$ (so that the sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are indeed well defined).
2. Show that
$$\forall n \in \mathbb{N},\ u_n < v_n.$$
3. Show that the sequence $(v_n)_{n\in\mathbb{N}}$ is decreasing.
4. Deduce that the sequence $(v_n)_{n\in\mathbb{N}}$ is convergent. We denote by ℓ the limit of $(v_n)_{n\in\mathbb{N}}$.
5. Deduce that the sequence $(u_n)_{n\in\mathbb{N}}$ converges to ℓ.
6. Show that the sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are adjacent sequences.
7. Show that
$$\forall n \in \mathbb{N},\ v_{n+1} - u_{n+1} = \frac{(v_n - u_n)^2}{2\left(\sqrt{v_n} + \sqrt{u_n}\right)^2}.$$
8. Show that
$$\forall n \in \mathbb{N},\ 0 < v_{n+1} - u_{n+1} < \frac{1}{8a}(v_n - u_n)^2.$$