SCAN 1 — S2 — Math Test #4 — 2h *March 24, 2017*

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

Exercise 1. For $n \in \mathbb{N}$ we define the function f_n as

$$f_n : \begin{array}{ccc} \mathbb{R}_+ & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \mathrm{e}^x + x^n - \mathrm{e}. \end{array}$$

1. Let $n \in \mathbb{N}$. Briefly justify that the function f_n is increasing, and justify that

$$\forall x \in (0,1),\ f_{n+1}(x) < f_n(x).$$

2. Let $n \in \mathbb{N}$. Briefly explain why there exists a unique element $u_n \in \mathbb{R}_+$ such that

$$f_n(u_n) = 0.$$

Justify, moreover, that $u_n \in (0,1)$.

3. Show that the sequence $(u_n)_{n\in\mathbb{N}}$ is increasing. You may proceed by contradiction as follows: for $n \in \mathbb{N}$, assume that $u_{n+1} \leq u_n$, and use both results of Question 1 to obtain a contradiction.

4. Deduce that the sequence $(u_n)_{n\in\mathbb{N}}$ converges to a limit $\ell \in (0,1]$.

5. We now show that $\ell = 1$, by contradiction: assume that $\ell \in (0,1)$. Explain why this assumption implies that $\lim_{n\to+\infty} u_n^n = 0$ and that $\mathrm{e}^\ell = \mathrm{e}$, hence a contradiction.

6. We now define the sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ as

$$\forall n \in \mathbb{N},\ \varepsilon_n = 1 - u_n.$$

From the previous questions we know that $\forall n \in \mathbb{N}$, $\varepsilon_n \in (0,1)$, and that $\varepsilon_n \underset{n\to+\infty}{\longrightarrow} 0$.

a) Explain why

$$\forall n \in \mathbb{N},\ \mathrm{e}^{-\varepsilon_n} - 1 = -\frac{(1-\varepsilon_n)^n}{\mathrm{e}}$$

and deduce that

$$\varepsilon_n \underset{n\to+\infty}{\sim} \frac{(1-\varepsilon_n)^n}{\mathrm{e}}.$$

b) Explain why

$$\forall n \in \mathbb{N},\ n\ln(1-\varepsilon_n) = 1 + \ln\left(\frac{1-\mathrm{e}^{-\varepsilon_n}}{\varepsilon_n}\right) + \ln(\varepsilon_n),$$

and show that

$$-n\varepsilon_n \underset{n\to+\infty}{\sim} \ln(\varepsilon_n).$$

Deduce that $n\varepsilon_n \underset{n\to+\infty}{\longrightarrow} +\infty$.

c) Show that

$$\ln\big(-\ln(\varepsilon_n)\big) \underset{n\to+\infty}{\sim} \ln(n) + \ln(\varepsilon_n),$$

and deduce that

$$-\ln(\varepsilon_n) \underset{n\to+\infty}{\sim} \ln(n).$$

d) Deduce that:

$$\varepsilon_n \underset{n\to+\infty}{\sim} -\frac{\ln(n)}{n}.$$

Exercise 2.

1. Use equivalents to determine the value of the following limit:

$$\lim_{x\to 0}\big(\cos(x)\big)^{1/x^2}.$$

2. Determine the simplest equivalent of the following expressions at the specified points:

$$(1)\ \sqrt{1+x^2}-x,\ \text{at } +\infty, \qquad (2)\ \ln\big(\cosh(x)\big),\ \text{at } +\infty, \qquad (3)\ \left(x+2-\frac{1}{x}\right)\arctan(x),\ \text{at } 0 \text{ and at } +\infty.$$

Exercise 3. The goal of this exercise is to compute a numerical value of $\cos(1/4)$.

1. Show that

$$1-\frac{1}{4^2 2!}+\frac{1}{4^4 4!}-\frac{1}{4^6 6!}<\cos(1/4)<1-\frac{1}{4^2 2!}+\frac{1}{4^4 4!}.$$

2. You're given:

$$1-\frac{1}{4^2 2!}+\frac{1}{4^4 4!}=\frac{5953}{6144}=0.96891276041\overline{6}, \qquad 1-\frac{1}{4^2 2!}+\frac{1}{4^4 4!}-\frac{1}{4^6 6!}=\frac{2857439}{2949120}=0.9689124213324652\overline{7},$$

(where the bar over the digits means that these digits are infinitely repeated). Deduce the value of $\cos(1/4)$ correct to as many decimal places as possible.

Exercise 4.

1. Let I be a neighborhood of 0 and let $f : I \longrightarrow \mathbb{R}$ be a continuous function that possesses a Taylor–Young expansion at 0 of the form

$$f(x) \underset{x\to 0}{=} a+bx+cx^n+o\big(x^n\big)$$

for some $n\in\mathbb{N}$ with $n\geq 2$ and $a,b\in\mathbb{R}$ and $c\in\mathbb{R}^*$.

a) Explain why f is differentiable at 0. Give an equation of the tangent to the graph of f at the point $\big(0,f(0)\big)$.

b) Show that there exists a neighborhood V of 0 such that

$$\forall x\in V\setminus\{0\},\ f(x)-a-bx \text{ and } cx^n \text{ have the same sign.}$$

2. We define the function[1]

$$f\ :\ \begin{array}{ccc}(-3\pi/2,0)\cup(0,1) & \longrightarrow & \mathbb{R}\\ x & \longmapsto & \dfrac{\ln(1-x)}{\sin(x)+\cos(x)-1}.\end{array}$$

Show that f can be extended by continuity at 0. The extension by continuity of f is still denoted by f.

3. Determine the third order Taylor–Young expansion of f at 0.

4. Give an equation of the tangent line Δ to the graph of f at $\big(0,f(0)\big)$, and specify the relative position of the graph of f with respect to Δ in a neighborhood of 0 (that is, specify whether the graph of f lies above or below, or crosses Δ in a neighborhood of 0). Sketch, on the same figure, the graph of f and Δ in a neighborhood of 0.

5. We define the function g by

$$g\ :\ \begin{array}{ccc}(-3\pi/2,1) & \longrightarrow & \mathbb{R}\\ x & \longmapsto & x^2+f(x).\end{array}$$

What is the third order Taylor–Young expansion of g at 0? (no justifications required). Give the relative position of the graph of g with respect to Δ in a neighborhood of 0. Sketch, on the same figure, the graph of g and Δ in a neighborhood of 0.

[1] you don't have to justify that f is well-defined.