INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — S1 — Math Test #1 — 1h30

October 20, 2017

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

Exercise 1. The goal of this exercise is to find a general formula for the linearization of $\cos^n$ where n is an even integer.

1. Let $x \in \mathbb{R}$. Show that
$$\cos^4(x) = \frac{1}{8}\big(\cos(4x) + 4\cos(2x) + 3\big).$$
2. Let $p \in \mathbb{N}^*$ and $x \in \mathbb{R}$.
   a) Show that
$$\cos^{2p}(x) = \frac{1}{2^{2p}} \sum_{k=0}^{2p} \binom{2p}{k} e^{2i(p-k)x}.$$
   b) Use an inversion of order of the sum to show that:
$$\sum_{k=p+1}^{2p} \binom{2p}{k} e^{2i(p-k)x} = \sum_{k=0}^{p-1} \binom{2p}{k} e^{2i(k-p)x}.$$
   c) Deduce that
$$\cos^{2p}(x) = \frac{1}{2^{2p-1}} \left(\sum_{k=0}^{p-1} \binom{2p}{k} \cos\big(2(p-k)x\big) + \frac{1}{2}\binom{2p}{p} \right).$$

Exercise 2. Let f be the polynomial function defined by
$$f : \begin{array}{rcl} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & x^5 - x^4 - 9x^3 + 5x^2 + 16x - 12. \end{array}$$

Show that:

- 1 is a root of f of multiplicity 2,
- -2 is a root of f of multiplicity 2,

and determine all the roots of f and their multiplicities, as well as the the factored form of f.

Exercise 3. Let f be the function defined by
$$f : \begin{array}{rcl} [-1,1] & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \begin{cases} x+1 & \text{if } x \in [-1,0) \\ -1 & \text{if } x = 0 \\ \dfrac{1}{x} & \text{if } x \in (0,1]. \end{cases} \end{array}$$

1. Plot the graph of f.
2. Determine graphically whether f is injective or not.
3. Determine (no justifications required):

$$f\big([-1,1]\big), \quad f\big([-1,0]\big), \quad f\big([0,1]\big), \quad f\big((0,1)\big),$$
$$f^{[-1]}(\mathbb{R}_+), \quad f^{[-1]}(\mathbb{R}_-), \quad f^{[-1]}\big([0,1]\big), \quad f^{[-1]}\big(\{1\}\big).$$

Exercise 4. Find all the real numbers x that satisfy

$$\cos(2x) = \sin(x).$$

Exercise 5. Let A, B and C be three non-empty sets, and let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be two surjective functions. Show that $g \circ f$ is surjective.

Exercise 6. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of real numbers that satisfies

$$\forall n \in \mathbb{N},\ u_{n+2} = \frac{3}{2}u_{n+1} - \frac{1}{2}u_n.$$

We define the sequence $(w_n)_{n\in\mathbb{N}}$ as follows:

$$\forall n \in \mathbb{N},\ w_n = u_{n+1} - u_n.$$

1. Show that the sequence $(w_n)_{n\in\mathbb{N}}$ is a geometric sequence of ratio $1/2$.
2. Deduce that
$$\forall n \in \mathbb{N},\ u_{n+1} = u_n + \frac{u_1 - u_0}{2^n}.$$
3. Let $n \in \mathbb{N}$. Compute the value of the following sum
$$\sum_{k=0}^{n}(u_{k+1} - u_k)$$
using two different methods[1] and deduce an explicit form for u_n (that only depends on n, u_0 and u_1).
4. Determine the variations of $(u_n)_{n\in\mathbb{N}}$ in terms of u_0 and u_1.

Exercise 7. Use the transformation of sums into products formula to show that sin is increasing on $[-\pi/2, \pi/2]$.

[1] one method is to use a telescopic sum, and the other method is to use the result of the previous question