INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — S1 — Math Test #2 — 1h30

December 1, 2017

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

Exercise 1. Let $\omega \in \mathbb{R}_+^*$.

1. Give the general real solution of the differential equation

$$(*) \qquad f'' + 4f' + \left(4 + \omega^2\right) f = 1.$$

2. Let f be a solution of $(*)$. Determine (if it exists) the value of the following limit:

$$\ell = \lim_{t \to +\infty} f(t).$$

Exercise 2. Show that there exists $x \in \mathbb{R}$ such that

$$x^2 + \mathrm{e}^{-x} = 4.$$

Exercise 3. Use the half angle formula to determine the value of the following limit:

$$\ell = \lim_{x \to 0} \frac{\cos(x) - 1}{x^2}.$$

Exercise 4. Let f be the function defined by

$$f : [0,2] \longrightarrow \mathbb{R}, \quad x \longmapsto \begin{cases} -2x + 1 & \text{if } x \in [0,1) \\ 3 & \text{if } x = 1 \\ x^2 - 1 & \text{if } x \in (1,2]. \end{cases}$$

1. Sketch the graph of f.
2. Determine, if they exist, the following elements. No justifications required. If they don't exist, write "DNE."

$$A = \inf f, \quad B = \sup f, \quad C = \min f, \quad D = \max f, \quad E = \inf_{[1,2]} f, \quad F = \min_{[1,2]} f, \quad G = \sup_{[0,1]} f, \quad H = \max_{[0,1]} f.$$

Exercise 5. Determine the value of the following limit:

$$\lim_{x \to 1} \frac{4}{x^3 + x^2 - x - 1} - \frac{3}{x^2 + x - 2}.$$

Exercise 6. We define the function f as

$$f \,:\, \begin{array}{ccc} (-1,+\infty) & \longrightarrow & \mathbb{R} \\ x & \longmapsto & (x+1)^{\frac{x+2}{x+1}}. \end{array}$$

1. Briefly explain why f is well-defined.
2. Determine the value of the following limits:

$$\ell_1 = \lim_{x\to+\infty} f(x) \qquad \text{and} \qquad \ell_2 = \lim_{x\to-1^+} f(x).$$

3. Show that

$$\lim_{x\to+\infty} \frac{f(x)}{x+1} = 1.$$

4. a) Let $x \in (-1,+\infty)$. Show that

$$f(x) - (x+1) = \ln(x+1)\frac{\exp\left(\dfrac{\ln(x+1)}{x+1}\right) - 1}{\dfrac{\ln(x+1)}{x+1}}.$$

   b) Deduce that

$$\lim_{x\to+\infty} \big(f(x) - (x+1)\big) = +\infty.$$

Exercise 7.

1. Show that

$$\forall x \in \mathbb{R},\ \cosh(x) \leq \cosh(2x).$$

2. Determine the value of the following limits (no justifications required):

$$\ell_1 = \lim_{x\to 0} \cosh(x) \qquad \text{and} \qquad \ell_2 = \lim_{x\to+\infty} \cosh(x).$$

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

$$\forall x \in \mathbb{R},\ \cos(x) + \cosh(x) \leq f(x) \leq \cos(x) + \cosh(2x).$$

   a) Determine the value of $f(0)$, and show that f is continuous at 0.
   b) Determine the value of the following limit:

$$\ell' = \lim_{x\to+\infty} f(x).$$

Exercise 8.

1. Determine the largest subset D of $\mathbb{R}$ such that for all $x \in D$ the following expressions are well-defined:

$$\tanh\left(\operatorname{arccosh}\left(\frac{1}{x}\right)\right) \qquad \text{and} \qquad \sqrt{1-x^2}.$$

2. Show that

$$\forall x \in D,\ \tanh\left(\operatorname{arccosh}\left(\frac{1}{x}\right)\right) = \sqrt{1-x^2}.$$

3. Deduce the solutions of the following equation in x:

$$\operatorname{arctanh}(x) = \operatorname{arccosh}\left(\frac{1}{x}\right).$$