SCAN 1 — S1 — Math Test #3 — 1h30 *December 22, 2017*

No documents, no calculators, no cell phones or electronic devices allowed. No free variables allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

Exercise 1 (*Viète's Formula*)**.** The goal of this exercise is to prove Viète's Formula, that is (informally):

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2} \cdots$$

We hence define the sequence $(u_n)_{n\geq 1}$ as

$$\begin{cases} u_1 = \sqrt{2} \\ \forall n \in \mathbb{N}^*,\ u_{n+1} = \sqrt{2+u_n}. \end{cases}$$

More formally, Viète's Formula reads as:

$$\frac{2}{\pi} = \lim_{N\to+\infty} \prod_{n=1}^{N} \frac{u_n}{2}.$$

1. Show that
$$\forall n \in \mathbb{N}^*,\ 0 < u_n < 2,$$
so that the sequence $(u_n)_{n\geq 1}$ is well defined.
2. Show that the sequence $(u_n)_{n\geq 1}$ converges and determine the value of its limit ℓ.
3. Show by induction (and using the double angle formula for sin) that
$$(*) \qquad \forall N \in \mathbb{N}^*,\ \forall x \in \mathbb{R},\ \sin(x) = 2^N \sin\left(\frac{x}{2^N}\right) \prod_{n=1}^{N} \cos\left(\frac{x}{2^n}\right).$$
4. Show that
$$\forall n \in \mathbb{N}^*,\ \cos\left(\frac{\pi}{2^{n+1}}\right) = \frac{u_n}{2}.$$
5. Deduce Viète's Formula. Hint: you will use Proposition $(*)$ with $x = \pi/2$ and the previous question.

Exercise 2 (*Lambert W function*)**.** We define the function

$$\begin{aligned} f\ :\ \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto x\mathrm{e}^x. \end{aligned}$$

1. Compute the derivative of f and deduce that f is increasing on $[-1,+\infty)$.
2. Explain why the function
$$\begin{aligned} g\ :\ [-1,+\infty) &\longrightarrow [-1/\mathrm{e},+\infty) \\ x &\longmapsto x\mathrm{e}^x \end{aligned}$$
is well defined and is a bijection.
3. We define $W = g^{-1}$.
   a) Show that W is differentiable on $(-1/\mathrm{e},+\infty)$. Is W differentiable at $-1/\mathrm{e}$?
   b) Determine the value of $W'(0)$.

Exercise 3 (*A funny equality*).

1. Preliminary question: for each of the following functions, recall their domain and range; also recall where they are differentiable and give an expression of their derivative:
$$\arcsin, \qquad \arctan, \qquad \sinh, \qquad \tanh\,.$$
No justifications required.

2. Determine the maximal subset D of $\mathbb{R}$ such that for all $x \in D$, both expressions
$$\arcsin\big(\tanh(x)\big)$$
and
$$\arctan\big(\sinh(x)\big)$$
are well-defined.

3. We define the functions
$$\begin{array}{rrcl} f : & D & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \arcsin\big(\tanh(x)\big), \end{array} \qquad \begin{array}{rrcl} g : & D & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \arctan\big(\sinh(x)\big), \end{array} \qquad \begin{array}{rrcl} u : & D & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & f(x) - g(x). \end{array}$$
   a) Show that f is differentiable on D and determine an explicit expression of f'. Simplify your answer as much as possible.
   b) Show that g is differentiable on D and determine an explicit expression of g'. Simplify your answer as much as possible.
   c) Determine an explicit expression of u' on D.
   d) What can you conclude about u? what can you conclude about f and g?

Exercise 4 (*Random questions*). The questions of this exercise are independent of each other.

1. Use the Mean Value Theorem to show that
$$\forall x \in \mathbb{R}_+^*,\ \mathrm{e}^x > 1 + x.$$

2. We define the function f as
$$\begin{array}{rrcl} f : & \mathbb{R}_+ & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & x^x. \end{array}$$
   a) Show that f is differentiable on $\mathbb{R}_+^*$ and for $x \in \mathbb{R}_+^*$ determine $f'(x)$ explicitly.
   b) Is f differentiable (from the right) at 0?

3. Let
$$\begin{array}{rrcl} g : & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \begin{cases} \sqrt{|x|}\sin(x)\cos\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases} \end{array}$$
Show that g is differentiable at 0 and determine the value of $g'(0)$.

Exercise 5 (*Darboux's Theorem*). Let $a, b \in \mathbb{R}$ such that $a < b$. Let $f : [a, b] \to \mathbb{R}$ be a differentiable function such that

- $f(a) < f(b)$,
- $f'(a) < 0 < f'(b)$.

The goal of this exercise is to prove that there exists $c \in (a, b)$ such that $f'(c) = 0$.
We define the function g as
$$\begin{array}{rrcl} g : & [a,b] & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \begin{cases} \dfrac{f(x) - f(a)}{x - a} & \text{if } x \neq a \\ f'(a) & \text{if } x = a. \end{cases} \end{array}$$

1. Show that g is continuous.
2. Show that there exists $p \in (a, b)$ such that $g(p) = 0$. What is the value of $f(p)$?
3. Show that there exists $c \in (a, p)$ such that $f'(c) = 0$. Hint: use Rolle's Theorem.