SCAN 1 — S1— Solution of Math Test #1

Romaric Pujol, romaric.pujol@insa-lyon.fr

November 7, 2018

Exercise 1.

1. See Figure 1.

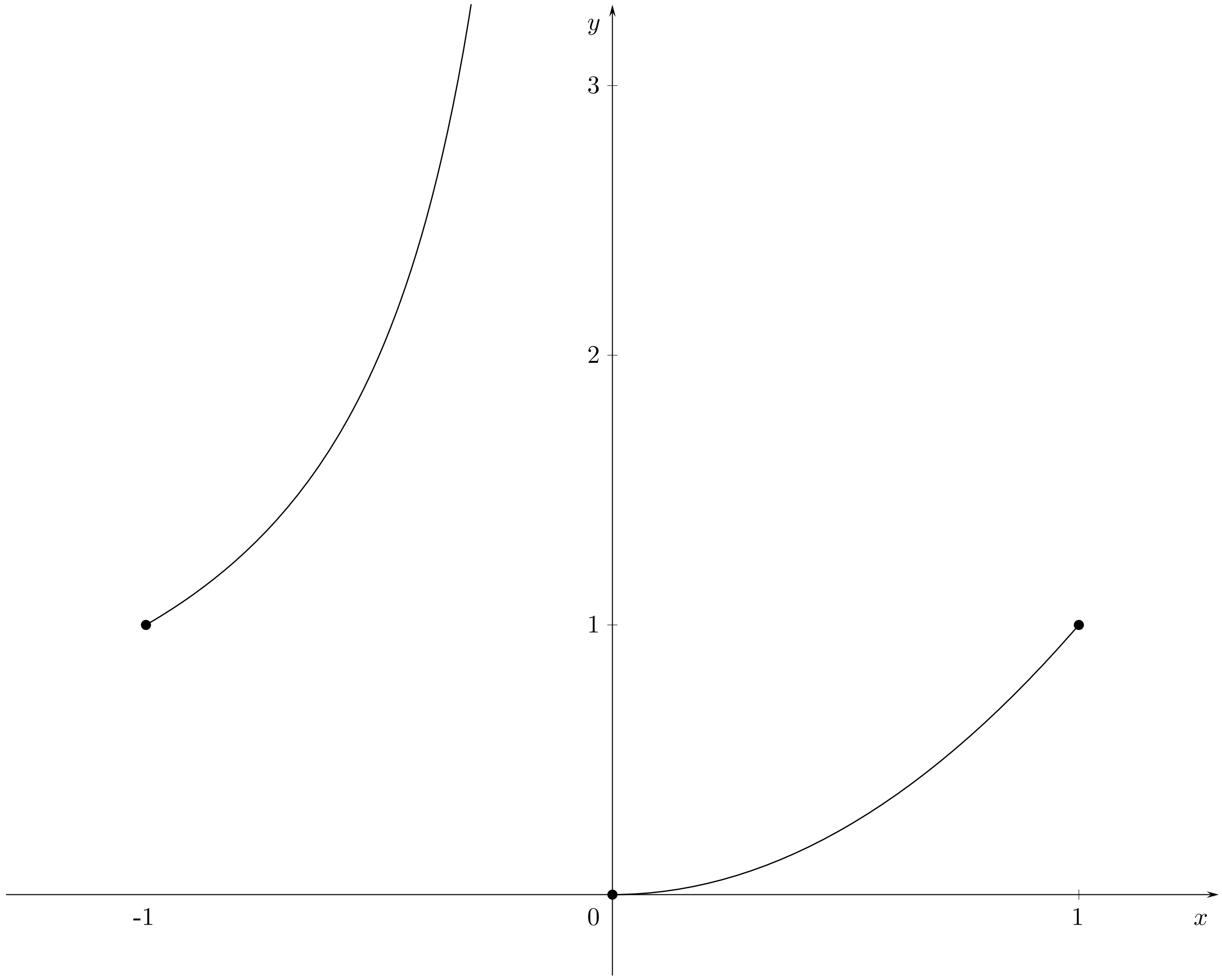

Figure 1 – Graph of function f of Exercise 1.

2. f is not injective: the horizontal line $y = 1$ intersects the graph of f twice.

3. f is surjective: if we project the points of the graph of f on the y-axis, we obtain all the non-negative numbers: $f\big([-1,1]\big) = \mathbb{R}_+ =$ codomain of f.

4.

$$f\big([0,1]\big) = [0,1], \qquad f\big([-1,0]\big) = [1,+\infty),$$
$$f\big([-1,0]\big) = \{0\} \cup [1,+\infty), \qquad f\big((-1,0)\cup(0,1)\big) = (0,1)\cup(1,+\infty),$$
$$f^{[-1]}\big([1,+\infty)\big) = [-1,0)\cup\{1\}, \qquad f^{[-1]}\big((1,+\infty)\big) = (-1,0),$$
$$f^{[-1]}\big([0,1]\big) = \{-1\}\cup[0,1], \qquad f^{[-1]}\big([0,1)\big) = [0,1).$$

Exercise 2.

Let $p \in \mathbb{N}$ and $k \in \mathbb{N}^*$.

$$\begin{aligned}\binom{p+k+1}{p+1} - \binom{p+k}{p+1} &= \frac{(p+k+1)!}{(p+1)!k!} - \frac{(p+k)!}{(p+1)!(k-1)!} \\ &= \frac{(p+k+1)! - (p+k)!k!}{(p+1)!k} \\ &= \frac{(p+k)!\big((p+k+1)-k\big)}{(p+1)!k!} \\ &= \frac{(p+k)!(p+1)}{(p+1)!k!} \\ &= \frac{(p+k)!}{p!k!} \\ &= \binom{p+k}{p}.\end{aligned}$$

Let $n, p \in \mathbb{N}$ such that $n > p$.

a) For $k \in \mathbb{N}^*$ define

$$x_k = \binom{p+k+1}{p+1}.$$

The sum we're trying to compute is:

$$\sum_{k=1}^{n-p} (x_{k+1} - x_k).$$

We recognize a telescopic sum, and we hence obtain:

$$\sum_{k=1}^{n-p} (x_{k+1} - x_k) = x_{n-p+1} - x_1 = \binom{n+1}{p+1} - \binom{p+1}{p+1} = \binom{n+1}{p+1} - 1.$$

b)

$$\begin{aligned}\sum_{k=0}^{n-p} \binom{p+k}{p} &= \binom{p}{p} + \sum_{k=1}^{n-p}\left(\binom{p+k+1}{p+1} - \binom{p+k}{p+1}\right) && \textit{by Question 1} \\ &= 1 + \binom{n+1}{p+1} - 1 && \textit{by the previous question} \\ &= \binom{n+1}{p+1}.\end{aligned}$$

In the case $p = n$, the left hand side of $(*)$ is:

$$\sum_{k=0}^{0} \binom{p+k}{p} = \binom{p}{p} = 1$$

and the right hand side of $(*)$ is:

$$\binom{n+1}{n+1} = 1.$$

Hence Equality $(*)$ is still true when $p = n$.

In the special case $p = 1$: applying the result of Question 2 yields

$$\forall n \in \mathbb{N}^*, \ \sum_{k=0}^{n-1} \binom{k+1}{1} = \binom{n+1}{2}.$$

Now,

$$\forall k \in \mathbb{N}, \ \binom{k+1}{1} = \frac{(k+1)!}{1!k!} = (k+1),$$

and
$$\binom{n+1}{2} = \frac{(n+1)!}{2!(n-1)!} = \frac{n(n+1)}{2}.$$
Hence we obtain:
$$\forall n \in \mathbb{N}^*,\ \sum_{k=0}^{n-1}(k+1) = \frac{n(n+1)}{2}.$$
Now the sum that appears on the left hand side can be written (using a shift of index) as
$$\sum_{k=0}^{n-1}(k+1) = \sum_{k=1}^{n} k,$$
and this sum is well-known to be equal to $n(n+1)/2$.
Let $n \in \mathbb{N}$ with $n \geq 2$. From the result of Question 2 we conclude:
$$\sum_{k=0}^{n-2}\binom{k+2}{2} = \binom{n+1}{3}.$$
Now,
$$\forall k \in \mathbb{N},\ \binom{k+2}{2} = \frac{(k+2)!}{2!k!} = \frac{(k+1)(k+2)}{2},$$
and
$$\binom{n+1}{3} = \frac{(n+1)!}{3!(n-2)!} = \frac{(n-1)n(n+1)}{6}.$$
Hence
$$\sum_{k=0}^{n-2}(k+1)(k+2) = \frac{(n-1)n(n+1)}{3},$$
and using a shift of index,
$$\sum_{k=0}^{n-2}(k+1)(k+2) = \sum_{k=2}^{n} k(k-1) = \sum_{k=1}^{n} k(k-1) = \sum_{k=1}^{n}(k^2-k) = \frac{(n-1)n(n+1)}{3},$$
hence
$$\begin{aligned}\sum_{k=1}^{n} k^2 &= \frac{(n-1)n(n+1)}{3} + \sum_{k=1}^{n} k \\ &= \frac{(n-1)n(n+1)}{3} + \frac{n(n+1)}{2} \\ &= \frac{n(n+1)\big(2(n-1)+3\big)}{6} \\ &= \frac{n(n+1)(2n+1)}{6}.\end{aligned}$$
Notice that this formula is also valid in the case $n = 1$, so that we finally conclude:
$$\forall n \in \mathbb{N}^*,\ S_n = \frac{(n+1)n(2n+1)}{6}.$$

Exercise 3. Let $x \in \mathbb{R}$. Then
$$\begin{aligned}\sin(3x) = \cos(x) &\iff \cos\left(3x - \frac{\pi}{2}\right) = \cos(x) \\ &\iff \exists k \in \mathbb{Z},\ \left(3x - \frac{\pi}{2} = x + 2k\pi \text{ or } 3x - \frac{\pi}{2} = -x + 2k\pi\right) \\ &\iff \exists k \in \mathbb{Z},\ \left(2x = \frac{\pi}{2} + 2k\pi \text{ or } 4x = \frac{\pi}{2} + 2k\pi\right) \\ &\iff \exists k \in \mathbb{Z},\ \left(x = \frac{\pi}{4} + k\pi \text{ or } x = \frac{\pi}{8} + k\frac{\pi}{2}\right)\end{aligned}$$

We now select the solutions that lie in $[0, 2\pi]$ and we obtain: the set of solutions is:

$$\left\{ \frac{\pi}{4}, \frac{5\pi}{4}, \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8} \right\}.$$

Exercise 4. Let $x, y \in B$ such that $g(x) = g(y)$. We need to show that $x = y$.
Since $x, y \in B$ and since f is surjective, there exists $a, b \in A$ such that $f(a) = x$ and $f(b) = y$. We then have:

$$(g \circ f)(a) = g(x) = g(y) = (g \circ f)(b).$$

Since $g \circ f$ is injective, we conclude that $a = b$. Hence

$$x = f(a) = f(b) = y$$

and indeed $x = y$.

Exercise 5.

1. Clearly, $f(1) = 0$, hence 1 is a root of f. We perform the following long division:

$$
\begin{array}{r|l}
 & x^4 - 4x^3 + 2x^2 + 4x - 3 \\ \hline
x - 1 & x^5 - 5x^4 + 6x^3 + 2x^2 - 7x + 3 \\
 & -\left(x^5 - x^4\right) \\ \hline
 & -4x^4 + 6x^3 + 2x^2 - 7x + 3 \\
 & -\left(-4x^4 + 4x^3\right) \\ \hline
 & 2x^3 + 2x^2 - 7x + 3 \\
 & -\left(2x^3 - 2x^2\right) \\ \hline
 & 4x^2 - 7x + 3 \\
 & -\left(4x^2 - 4x\right) \\ \hline
 & -3x + 3 \\
 & -\left(-3x + 3\right) \\ \hline
 & 0
\end{array}
$$

   We denote by g_1 the quotient:

$$\forall x \in \mathbb{R},\ g_1(x) = x^4 - 4x^3 + 2x^2 + 4x - 3.$$

   Now, $g_1(1) = 0$, so we divide $g_1(x)$ by $x - 1$:

$$
\begin{array}{r|l}
 & x^3 - 3x^2 - x + 3 \\ \hline
x - 1 & x^4 - 4x^3 + 2x^2 + 4x - 3 \\
 & -\left(x^4 - x^3\right) \\ \hline
 & -3x^3 + 2x^2 + 4x - 3 \\
 & -\left(-3x^3 + 3x^2\right) \\ \hline
 & -x^2 + 4x - 3 \\
 & -\left(-x^2 + x\right) \\ \hline
 & 3x - 3 \\
 & -\left(3x - 3\right) \\ \hline
 & 0
\end{array}
$$

   We denote by g_2 the quotient:

$$\forall x \in \mathbb{R},\ g_2(x) = x^3 - 3x^2 - x + 3.$$

Now, $g_2(1) = 0$, so we divide $g_2(x)$ by $(x-1)$:

$$
\begin{array}{r|l}
 & x^2 - 2x - 3 \\
\hline
x - 1 & x^3 - 3x^2 - x + 3 \\
 & -\left(x^3 - x^2\right) \\
 & -2x^2 - x + 3 \\
 & -\left(-2x^2 + 2x\right) \\
 & -3x + 3 \\
 & -\left(-3x + 3\right) \\
 & 0
\end{array}
$$

We denote by g_3 the quotient:

$$\forall x \in \mathbb{R},\ g_3(x) = x^2 - 2x - 3.$$

Now, $g_3(1) \neq 0$, so the multiplicity of 1 is 3.

2. At this point we have:

$$\forall x \in \mathbb{R},\ f(x) = (x-1)^3\left(x^2 - 2x - 3\right).$$

Hence the other roots of f are that of g_3, namely 3 and -1, both of multiplicity 1.

3. The real factored form of f is hence:

$$\forall x \in \mathbb{R},\ f(x) = (x-1)^3(x+1)(x-3).$$

Exercise 6. We notice that the sequence $(u_n)_{n\geq 1}$ only has positive terms, hence we can use the quotient of two consecutive terms to determine its variations.
Let $n \in \mathbb{N}^*$. Then:

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)!}{2^{n+1}}\frac{2^n}{n!} = \frac{(n+1)}{2} \geq 1.$$

Hence the sequence $(u_n)_{n\geq 1}$ is non-decreasing.

Exercise 7. Let $h \in [-1, +\infty)$. For $n \in \mathbb{N}$ we denote by (P_n) the following proposition:

(P_n) $$(1+h)^n \geq 1 + nh.$$

- base case: (P_0) is obviously true, as (P_0) reads as:

$$(1+h)^0 \geq 1 + 0h.$$

- inductive step: assume that (P_n) is true for some $n \in \mathbb{N}$. By the inductive hypothesis (P_n) we have: $(1+h)^n \geq 1 + nh$. Since $1 + h \geq 0$ (very important hypothesis here!), we can multiply the inequality (P_n) by the non-negative number $1 + h$ and we obtain:

$$(1+h)(1+h)^n =\geq (1+h)(1+nh) = 1 + nh + h + nh^2 = 1 + (n+1)h + nh^2 \geq 1 + (n+1)h,$$

hence

$$(1+h)^{n+1} \geq 1 + (n+1)h,$$

hence (P_{n+1}) is true.