INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — S1 — Math Test #1 — 1h30 *November 7, 2018*

No documents, no calculators, no cell phones or electronic devices allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

Exercise 1. Let f be the function defined by

$$f : [-1,1] \longrightarrow \mathbb{R}_+ \qquad x \longmapsto \begin{cases} x^2 & \text{if } x \in [0,1] \\ -\dfrac{1}{x} & \text{if } x \in [-1,0). \end{cases}$$

1. Plot the graph of f.
2. Determine graphically whether f is injective or not.
3. Determine graphically whether f is surjective or not.
4. Determine the following sets (no justifications required):

$$f\big([0,1]\big), \quad f\big([-1,0]\big), \quad f\big([-1,0)\big), \quad f\big((-1,0)\cup(0,1)\big),$$
$$f^{[-1]}\big([1,+\infty)\big), \quad f^{[-1]}\big((1,+\infty)\big), \quad f^{[-1]}\big([0,1]\big), \quad f^{[-1]}\big([0,1)\big).$$

Exercise 2.

1. Show that
$$\forall p \in \mathbb{N},\ \forall k \in \mathbb{N}^*,\ \binom{p+k}{p} = \binom{p+k+1}{p+1} - \binom{p+k}{p+1}.$$
2. Let $n, p \in \mathbb{N}$ such that $n > p$.
   a) Simplify the following sum:
   $$\sum_{k=1}^{n-p}\left(\binom{p+k+1}{p+1} - \binom{p+k}{p+1}\right).$$
   (notice that $n-p \geq 1$ so that the sum is well-defined).
   Hint: You may use a shift of index, or recognize a telescopic sum.
   b) Deduce that
   $$(*) \qquad \sum_{k=0}^{n-p}\binom{p+k}{p} = \binom{n+1}{p+1}.$$
   Is the Equality $(*)$ still valid when $p = n$?
3. Explain why the special case $p = 1$ corresponds to the well-known formula for the sum of the first consecutive integers (sum of an arithmetic progression of common difference 1).
4. Use Equality $(*)$ in the special case $p = 2$ to determine, for $n \in \mathbb{N}^*$, the value of the following sum:
$$S_n = \sum_{k=1}^{n} k^2.$$

Exercise 3. Find all numbers $x \in [0, 2\pi]$ that satisfy

$$(*) \qquad \sin(3x) = \cos(x).$$

Exercise 4. Let A, B and C be three non-empty sets, and let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be two functions. We assume that

- f is surjective,
- $g \circ f$ is injective.

Prove that g is injective.

Exercise 5. We define the following polynomial function:

$$\begin{array}{rccl} f : & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & x^5 - 5x^4 + 6x^3 + 2x^2 - 7x + 3. \end{array}$$

1. Show that 1 is a root of f and determine its multiplicity.
2. Deduce the other roots of f and their multiplicities.
3. Deduce the factored form of f in $\mathbb{R}$.

Exercise 6. Let $(u_n)_{n\geq 1}$ be the sequence defined by

$$\forall n \in \mathbb{N}^*,\ u_n = \frac{n!}{2^n}.$$

Determine the variations of the sequence $(u_n)_{n\geq 1}$.

Exercise 7 (*Bernoulli's Inequality*)**.** Let $h \in [-1, +\infty)$. Prove by induction that:

$$\forall n \in \mathbb{N},\ (1 + h)^n \geq 1 + nh.$$

What more direct proof can you give in the case $h \geq 0$ and $n \geq 1$?