INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — S2 — Math Test #4 — 2h *March 29, 2019*

No documents, no cell phones allowed. Calculators allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

Exercise 1.

1. Show that
$$\frac{1}{4}-\frac{1}{6}\left(\frac{1}{4}\right)^3+\frac{1}{120}\left(\frac{1}{4}\right)^5-\frac{1}{5040}\left(\frac{1}{4}\right)^7<\sin\left(\frac{1}{4}\right)<\frac{1}{4}-\frac{1}{6}\left(\frac{1}{4}\right)^3+\frac{1}{120}\left(\frac{1}{4}\right)^5.$$

2. Deduce the value of $\sin(1/4)$ correct to as many decimal places as possible.

You're given (in case your calculator is broken):

$$\frac{1}{4}-\frac{1}{6}\left(\frac{1}{4}\right)^3+\frac{1}{120}\left(\frac{1}{4}\right)^5-\frac{1}{5040}\left(\frac{1}{4}\right)^7=\frac{20429471}{82575360}=0.247403959244016617\,\overline{063492},$$
$$\frac{1}{4}-\frac{1}{6}\left(\frac{1}{4}\right)^3+\frac{1}{120}\left(\frac{1}{4}\right)^5=\frac{30401}{122880}=0.2474039713541\,\overline{6}.$$

Exercise 2. Define the function

$$f : \begin{array}{ccc}(-1,0)\cup(0,\pi) & \longrightarrow & \mathbb{R}\\ x & \longmapsto & \dfrac{1}{\sin x}-\dfrac{1}{\ln(1+x)}.\end{array}$$

1. Show that f possesses an extension by continuity $\tilde{f}$ at 0.
2. Show that $\tilde{f}$ is differentiable at 0 and determine the relative position of $\tilde{f}$ with respect to its tangent line Δ at 0 in a neighborhood of 0.
3. Sketch the graph of $\tilde{f}$ and Δ (in a neighborhood of 0), on the same figure.

Exercise 3 (*Equivalents*)**.** The questions of this exercise are independent from each other.

1. Determine the simplest equivalent of
$$\left(1+\frac{1}{n}\right)^n-\mathrm{e}$$
as $n\longrightarrow+\infty$.
2. Determine the simplest equivalent of
$$\left(1+x^x\right)^{1/x}-x$$
as $x\longrightarrow+\infty$.
3. Determine the simplest equivalent of
$$\ln\bigl(\cos(x)\bigr)+\frac{x^2}{2}$$
as $x\longrightarrow 0$.

Exercise 4. Use a long division to determine the 5-th order[4] Taylor–Young expansion of tan at 0. Deduce the value of the following limit:
$$\lim_{x\to0}\frac{\tan(x)-x-x^3/3}{(\cos(x)-1)\sin^3(x)}.$$

[4] 5-th order means with $o(x^5)$.

Exercise 5. Define the function:

$$f : \begin{array}{rcl} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & e^x \cos(x). \end{array}$$

1. Determine the largest open interval I of $\mathbb{R}$ containing 0 such that f is increasing on I.

   We hence denote by g the following bijection:

   $$g : \begin{array}{rcl} I & \longrightarrow & J \\ x & \longmapsto & f(x), \end{array}$$

   where $J = f(I)$.

2. Explain why g^{-1} is differentiable on J, and for $y \in J$ give an expression of $\left(g^{-1}\right)'(y)$.

   You're given that g^{-1} is of class C^∞ on J.

3. Give the 4-th order[5] Taylor–Young expansion of g at 0.

4. Explain why g^{-1} possesses a 4-th order Taylor–Young expansion at 1

   $$g^{-1}(y) \underset{y\to 1}{=} r + s(y-1) + t(y-1)^2 + u(y-1)^3 + v(y-1)^4 + o\big((y-1)^4\big),$$

   where r, s, t, u and v are real numbers (that you don't have to determine in this question).

5. What is the value of r and of s?

6. By substituting the expansion of g^{-1} in that of g, determine explicitly the 4-th order Taylor–Young expansion of g^{-1} at 1.

Exercise 6. Define the function

$$f : \begin{array}{rcl} \mathbb{R}_+^* & \longrightarrow & \mathbb{R} \\ x & \longmapsto & x\ln(x). \end{array}$$

We admit that for all $n \in \mathbb{N}^*$, there exists a unique real number $u_n \in [1/e, 1]$ such that $f(u_n) = -1/n$. We also admit that the sequence $(u_n)_{n\in\mathbb{N}^*}$ is increasing.

1. Show that the sequence $(u_n)_{n\in\mathbb{N}^*}$ converges to 1.

2. We define the sequence $(v_n)_{n\in\mathbb{N}^*}$ as:

   $$\forall n \in \mathbb{N}^*,\ v_n = u_n - 1.$$

   a) Show that

   $$u_n \ln(u_n) \underset{n\to+\infty}{\sim} v_n.$$

   b) Deduce that

   $$u_n \underset{n\to+\infty}{=} 1 - \frac{1}{n} + o\left(\frac{1}{n}\right).$$

3. Check that

   $$\forall n \in \mathbb{N}^*,\ u_n = \exp\left(-\frac{1}{nu_n}\right)$$

   and determine the real number a such that

   $$u_n = 1 - \frac{1}{n} + \frac{a}{n^2} + o\left(\frac{1}{n^2}\right).$$

Exercise 7. Compute the numerical approximation for the integral

$$I = \int_0^1 t^2\,dt,$$

corresponding the Riemann sum associated to the tagged subdivision $\mathscr{T} = \big((0,\ 0.5,\ 0.8,\ 1),\ (0.1,\ 0.7,\ 0.9)\big)$. Compute the exact value of I.

[5] 4-th order means with $o(x^4)$.