INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — S2— Solution of Math Test #5

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Exercise 1.

1.
$$I=\int_0^1\frac{\mathrm{e}^x}{1+\mathrm{e}^x}\,\mathrm{d}x=\Big[\ln\big(1+\mathrm{e}^x\big)\Big]_{x=0}^{x=1}=\ln(1+\mathrm{e})-\ln(2)=\ln\left(\frac{1+\mathrm{e}}{2}\right).$$

2.
$$J_0=\int_0^1\frac{1}{1+\mathrm{e}^x}\,\mathrm{d}x=\int_0^1\left(1-\frac{\mathrm{e}^x}{1+\mathrm{e}^x}\right)\mathrm{d}x=1-I=1-\ln\left(\frac{1+\mathrm{e}}{2}\right).$$

3. a) Let $n\in\mathbb{N}$. Then
$$J_{n+1}-J_n=\int_0^1\mathrm{e}^{-nx}\frac{\mathrm{e}^{-x}-1}{1+\mathrm{e}^x}\,\mathrm{d}x.$$
For $x\in[0,1]$
$$\mathrm{e}^{-nx}\frac{\mathrm{e}^{-x}-1}{1+\mathrm{e}^x}\leq 0$$
(since $\mathrm{e}^{-x}\leq 1$). Moreover, since the function we're integrating is continuous and not identically nil, and since the endpoints of the integral satisfy $0<1$, we conclude that $J_{n+1}-J_n<0$. Hence, the sequence $(J_n)_{n\in\mathbb{N}}$ is decreasing.

   b) Let $n\in\mathbb{N}^*$. Since the sequence $(J_n)_{n\in\mathbb{N}}$ is decreasing,
$$J_{n+1}\leq J_n\leq J_{n-1}.$$
Adding J_n to all terms of this inequality yields
$$J_{n+1}+J_n\leq 2J_n\leq J_n+J_{n-1},$$
and hence,
$$\frac{J_{n+1}+J_n}{2}\leq J_n\leq\frac{J_n+J_{n-1}}{2}.$$

4. Let $n\in\mathbb{N}^*$. Then
$$J_{n-1}+J_n=\int_0^1\frac{\mathrm{e}^{-(n-1)x}+\mathrm{e}^{-nx}}{1+\mathrm{e}^x}\,\mathrm{d}x=\int_0^1\mathrm{e}^{-nx}\,\mathrm{d}x=\frac{1-\mathrm{e}^{-n}}{n}.$$

5. From the two previous questions we conclude that:
$$\forall n\in\mathbb{N}^*,\ \frac{1-\mathrm{e}^{-n-1}}{2(n+1)}\leq J_n\leq\frac{1-\mathrm{e}^{-n}}{2n}.$$
i.e.,
$$\forall n\in\mathbb{N}^*,\ \big(1-\mathrm{e}^{-n-1}\big)\frac{n}{n+1}\leq 2nJ_n\leq 1-\mathrm{e}^{-n}.$$
Hence, by the Squeeze Theorem,
$$\lim_{n\to+\infty}2nJ_n=1,$$
as required.

Exercise 2.

1. a) Let $n\in\mathbb{N}$. Since $a>0$,
$$\forall x\in[0,1],\ x^n\leq x^n\mathrm{e}^{ax}\leq\mathrm{e}^a x^n,$$
hence (since the endpoints of the integral satisfy $0<1$):
$$\int_0^1 x^n\,\mathrm{d}x\leq I_n\leq\int_0^1\mathrm{e}^a x^n\,\mathrm{d}x$$
hence
$$\frac{1}{n+1}\leq I_n\leq\frac{\mathrm{e}^a}{n+1}.$$

b) By the Squeeze Theorem we conclude that $\lim_{n\to+\infty} I_n = 0$.

2. Let $n\in\mathbb{N}$. Then

$$I_{n+1} = \int_0^1 x^{n+1}\mathrm{e}^{ax}\,\mathrm{d}x = \left[x^{n+1}\frac{\mathrm{e}^{ax}}{a}\right]_{x=0}^{x=1} - \int_0^1 (n+1)x^n\frac{\mathrm{e}^{ax}}{a}\,\mathrm{d}x = \frac{\mathrm{e}^a}{a} - \frac{n+1}{a}I_n = \frac{1}{a}\big(\mathrm{e}^a - (n+1)I_n\big).$$

3. From the previous questions, we conclude that:

$$0 = \lim_{n\to+\infty} I_{n+1} = \lim_{n\to+\infty}\frac{1}{a}\big(\mathrm{e}^a-(n+1)I_n\big),$$

hence

$$\lim_{n\to+\infty}(n+1)I_n = \mathrm{e}^a,$$

hence

$$I_n \underset{n\to+\infty}{\sim} \frac{\mathrm{e}^a}{n+1} \underset{n\to+\infty}{\sim} \frac{\mathrm{e}^a}{n}.$$

Exercise 3. We recognize a rational function, with the degree of the numerator less than that of the denominator, and where the polynomial part X^2+2X+2 is irreducible in $\mathbb{R}$ (since the discriminant is $2^2-4\times 2=-4<0$), hence there exists $A,B,C\in\mathbb{R}$ such that:

$$\frac{X}{(X+1)\big(X^2+2X+2\big)} = \frac{A}{X+1} + \frac{BX+C}{X^2+2X+2}.$$

We now determine the values of A, B and C (there are other ways that what is presented here):

- for A, multiply by $X+1$ and evaluate at $X=-1$, and we get $A=-1$,
- for C, we evaluate at $X=0$, and we get $0=A+C/2$, hence (since $A=-1$), $C=2$,
- for B, we multiply by X and take the limit as $X\to+\infty$, and we get $0=A+B$, hence (since $A=-1$), $B=1$.

Hence,

$$\forall x\in(-1,+\infty),\ f(x) = \frac{-1}{x+1} + \frac{x+2}{x^2+2x+2}.$$

We now rewrite f as follows: for $x\in(-1,+\infty)$,

$$f(x) = \frac{-1}{x+1} + \frac{1}{2}\,\frac{2x+2}{x^2+2x+2} + \frac{1}{x^2+2x+2} = \frac{-1}{x+1} + \frac{1}{2}\,\frac{2x+2}{x^2+2x+2} + \frac{1}{(x+1)^2+1}.$$

Hence, an antiderivative of f is given by the following function F:

$$\begin{aligned} F\ :\ (-1,+\infty) &\longrightarrow \mathbb{R}\\ x &\longmapsto -\ln(x+1) + \frac{1}{2}\ln\big(x^2+2x+2\big) + \arctan(x+1).\end{aligned}$$

Exercise 4. The function $t\mapsto \mathrm{e}^{t^2}$ is continuous on $\mathbb{R}$ hence, for $x\in\mathbb{R}$, the integral defining g is well-defined. Let

$$\begin{aligned} f\ :\ \mathbb{R} &\longrightarrow \mathbb{R}\\ t &\longmapsto \mathrm{e}^{t^2},\end{aligned}$$

and let F be an antiderivative of f (such an antiderivative exists since f is continuous). Then,

$$\forall x\in\mathbb{R},\ g(x) = F\big(\cos(x)\big) - F\big(\sinh(x)\big).$$

Since F is differentiable we conclude, by the chain rule and elementary differentiation rules, that g is differentiable and that:

$$\forall x\in\mathbb{R},\ g'(x) = -\sin(x)F'\big(\cos(x)\big) - \cosh(x)F'\big(\sinh(x)\big) = -\sin(x)\mathrm{e}^{\cos^2(x)} - \cosh(x)\mathrm{e}^{\sinh^2(x)}.$$

Exercise 5.

$$\text{(S)} \underset{\substack{R_2 \leftarrow R_2 - aR_1 \\ R_3 \leftarrow R_3 - aR_1}}{\iff} \begin{cases} x + y + z = 1 - a \\ (1-2a)y + (1-2a)z = -a + 2a^2 \\ y + z = 0 \end{cases}$$

$$\underset{R_2 \leftrightarrow R_3}{\iff} \begin{cases} x + y + z = 1 - a \\ y + z = 0 \\ (1-2a)y + (1-2a)z = -a + 2a^2 \end{cases}$$

$$\underset{R_3 \leftarrow R_3 - (1-2a)R_2}{\iff} \begin{cases} x + y + z = 1 - a \\ y + z = 0 \\ 0 = -a + 2a^2 = a(-1+2a). \end{cases}$$

Hence the rank of System (S) is 2, and System (S) is compatible if and only if $a = 0$ or $a = 1/2$.

- If $a = 0$,

$$\text{(S)} \iff \begin{cases} x + y + z = 1 \\ y + z = 0 \end{cases} \iff \begin{cases} x = 1 \\ y = -z \\ z = z, \end{cases}$$

- if $a = 1/2$,

$$\text{(S)} \iff \begin{cases} x + y + z = 1/2 \\ y + z = 0 \end{cases} \iff \begin{cases} x = 1/2 \\ y = -z \\ z = z. \end{cases}$$

Exercise 6.

1. Clearly, $2u + v = (2,2,2,2) + (1,2,-1,2) = (3,4,1,4) = w$, hence $w \in \operatorname{Span}\{u, v\}$.

2. We conclude that $\operatorname{Span}\{u,v,w\} = \operatorname{Span}\{u,v\}$. Now the vectors u and v are clearly non-collinear, hence the family (u,v) is independent, hence $\operatorname{rk}\mathscr{F} = \dim \operatorname{Span}\{u,v\} = 2$.

   The vectors a and b are independent, hence $\operatorname{rk}\mathscr{G} = 2$.

3. a) We compute the rank of (u,v,a,b):

$$\operatorname{rk}(u,v,a,b) = \operatorname{rk}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 1 & 2 & 0 & -1 \end{pmatrix}$$

$$\underset{\substack{C_2 \leftarrow C_2 - C_1 \\ C_3 \leftarrow C_3 - C_1 \\ C_4 \leftarrow C_4 - C_1}}{=} \operatorname{rk}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -3 & 0 & -1 \\ 1 & 1 & -1 & -2 \end{pmatrix}$$

$$\underset{C_3 \leftarrow C_3 + C_1}{=} \operatorname{rk}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & -3 & -3 & -1 \\ 1 & 1 & 0 & -2 \end{pmatrix}$$

$$\underset{C_3 \leftrightarrow C_4}{=} \operatorname{rk}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & -3 & -1 & -3 \\ 1 & 1 & -2 & 0 \end{pmatrix}$$

$$\underset{C_4 \leftarrow C_4 - 3C_3}{=} \operatorname{rk}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & -3 & -1 & 0 \\ 1 & 1 & -2 & 6 \end{pmatrix} = 4.$$

   Hence $\dim(F+G) = \dim \operatorname{Span}\{u,v,a,b\} = 4$. By the Inclusion–Equality Theorem, we conclude that $F + G = E$.

b) By Grassmann's formula,

$$\dim(F \cap G) = \dim F + \dim G - \dim(F + G) = 2 + 2 - 4 = 0.$$

c) Since $\dim(F \cap G) = \{0_E\}$, the subspaces F and G are independent, hence the sum $F + G$ is a direct sum.

Exercise 7.

1. $\mathscr{B} = \big(1, X, X^2\big)$, and $\dim E = 3$.

2. a) Clearly, $0_E \in F$, hence $F \neq \emptyset$.
   Let $P, Q \in F$ and let $\lambda, \mu \in \mathbb{R}$. Then

   $$(\lambda P+\mu Q)(0)+(\lambda P+\mu Q)'(1) = \lambda P(0)+\mu Q(0)+\lambda P'(1)+\mu Q'(1) = \lambda\big(P(0)+P'(1)\big)+\mu\big(Q(0)+Q'(1)\big) = 0,$$

   since $P, Q \in F$, hence $\lambda P + \mu Q \in F$. We conclude that F is a subspace of E.

   b) Let $P \in G$, say $P = a + b(1 + X) + c\big(1 + X^2\big) = cX^2 + bX + (a + b + c)$ for some $a, b, c \in \mathbb{R}$. Then

   $$\begin{aligned} P \in F &\iff (a+b+c) + 2c + b = 0 \\ &\iff \big\{ a + 2b + 3c = 0 \\ &\iff \begin{cases} a = -2b - 3c \\ b = b \\ c = c. \end{cases} \\ &\iff P = (-2b - 3c) + b(1 + X) + c\big(1 + X^2\big) \\ &\iff P = b\big(-1 + X\big) + c\big(-2 + X^2\big). \end{aligned}$$

   Hence a basis of $F \cap G$ is $\big(-1 + X, -2 + X^2\big)$.

Exercise 8.

1. "$f : E \to F$ is a linear map" means:

   $$\forall u, v \in E,\ \forall \lambda \in \mathbb{K},\ f(u + \lambda v) = f(u) + \lambda f(v).$$

2. a) $\operatorname{Ker} f = \big\{u \in E \,\big|\, f(u) = 0_F\big\}$.

   b) Since f is linear, $f(0_E) = 0_F$, hence $0_E \in \operatorname{Ker} f$, hence $\operatorname{Ker} f \neq \emptyset$.
   Let $u, v \in \operatorname{Ker} f$ and let $\lambda \in \mathbb{K}$. Then, since f is linear, $f(u + \lambda v) = f(u) + \lambda f(v) = 0_F + \lambda 0_F = 0_F$, hence $u + \lambda v \in \operatorname{Ker} f$. Hence $\operatorname{Ker} f$ is a subspace of E.

   c) The Rank–Nullity Theorem for f is:

   $$\dim E = \dim \operatorname{Ker} f + \operatorname{rk} f.$$