SCAN 1 — S2 — Math Test #5 — 2h

May 17, 2019

No documents, no cell phones allowed. Calculators allowed. Cute and fluffy pets allowed (for moral support only).
All your answers must be fully (but concisely) justified, unless noted otherwise.

Exercise 1. We define the sequence $(J_n)_{n\in\mathbb{N}}$ as:

$$\forall n \in \mathbb{N},\ J_n = \int_0^1 \frac{e^{-nx}}{1+e^x}\,dx.$$

We also define:

$$I = \int_0^1 \frac{e^x}{1+e^x}\,dx.$$

1. Compute the value of I.
2. Find a relation between J_0 and I, and deduce the value of J_0.
3. a) Determine the variations of the sequence $(J_n)_{n\in\mathbb{N}}$.
   b) Deduce, without any further computations, that:
$$\forall n \in \mathbb{N}^*,\ \frac{J_n + J_{n+1}}{2} \le J_n \le \frac{J_{n-1}+J_n}{2}.$$
4. Compute, for $n \in \mathbb{N}^*$, the value of $J_{n-1} + J_n$.
5. Deduce that $J_n \underset{n\to+\infty}{\sim} \frac{1}{2n}$.

Exercise 2. Let $a \in \mathbb{R}_+^*$. We define the sequence $(I_n)_{n\in\mathbb{N}}$ as:

$$\forall n \in \mathbb{N},\ I_n = \int_0^1 x^n e^{ax}\,dx.$$

1. a) Show that:
$$\forall n \in \mathbb{N},\ \frac{1}{n+1} \le I_n \le \frac{e^a}{n+1},$$
   b) Deduce the value of $\lim\limits_{n\to+\infty} I_n$.
2. Let $n \in \mathbb{N}$. Use an integration by parts to find a relation between I_{n+1} and I_n.
3. Deduce from Questions 1b and 2 a simple equivalent of I_n as $n \longrightarrow +\infty$.

Exercise 3. Let

$$f : \begin{array}{ccc} (-1,+\infty) & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \dfrac{x}{(x+1)\big(x^2+2x+2\big)}. \end{array}$$

Determine an antiderivative of f.

Exercise 4. Define the function g as:

$$g : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \displaystyle\int_{\sinh(x)}^{\cos(x)} e^{t^2}\,dt. \end{array}$$

Explain why g is well-defined. Show that g is differentiable and determine an expression of g'.

Exercise 5. Let $a \in \mathbb{R}$. Solve the following linear system, and specify its rank:

$$(\mathrm{S}) \qquad \begin{cases} x + y + z = 1 - a \\ ax + (1-a)y + (1-a)z = a^2 \\ ax + (1+a)y + (1+a)z = a - a^2. \end{cases}$$

Exercise 6. Let $E = \mathbb{R}^4$ and define the following vectors of E:

$$u = (1,1,1,1), \qquad v = (1,2,-1,2), \qquad w = (3,4,1,4),$$
$$a = (1,0,1,0), \qquad b = (1,1,0,-1).$$

We define

$$F = \mathrm{Span}\{u, v, w\}, \qquad G = \mathrm{Span}\{a, b\}.$$

1. Show that $w \in \mathrm{Span}\{u, v\}$.
2. Determine the rank of the family $\mathscr{F} = (u, v, w)$ and of the family $\mathscr{G} = (a, b)$.
3. a) Show that $F + G = E$.
   b) Use Grassmann's formula to determine the dimension of $F \cap G$.
   c) Is the sum $F + G$ a direct sum?

Exercise 7. Let $E = \mathbb{R}_2[X]$ be the real vector space of formal polynomials with real coefficients, indeterminate X, and of degree non-greater than 2.

1. Recall (without any justifications) the standard basis $\mathscr{B}$ of E, as well as the dimension of E.
2. Let
$$F = \left\{ P \in E \,\middle|\, P(0) + P'(1) = 0 \right\},$$
and
$$G = \mathrm{Span}\left\{1, 1 + X, 1 + X^2\right\}.$$
   a) Show that F is a subspace of E.
   b) Determine a basis of $F \cap G$.

Exercise 8. Let E and F be two vector spaces over $\mathbb{K}$.

1. Let $f : E \to F$. Recall the definition of "f is a linear map."
2. Let $f : E \to F$ be a linear map.
   a) Recall the definition of $\mathrm{Ker}\, f$.
   b) Show that $\mathrm{Ker}\, f$ is a subspace of E.
   c) Recall the Rank–Nullity Theorem.