SCAN 1 — S2— Solution of Math Test #6

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Exercise 1.

1. a)

$$\operatorname{rk}(A-I_4)=\operatorname{rk}\begin{pmatrix}6&6&-6&-6\\4&4&-4&-4\\4&4&-4&-4\\8&8&-8&-7\end{pmatrix}=1.$$

b) We conclude that the matrix $A-I_4$ is not invertible, hence 1 is an eigenvalue of A. Moreover, by the Rank–Nullity Theorem,

$$\dim E_1=\dim\operatorname{Ker}(A-I_4)=4-\operatorname{rk}(A-I_4)=3.$$

We conclude that the multiplicity of 1 is *at least* 3.

c) We're missing one eigenvalue, so we can use the trace trick to determine it: we know that the trace of A is the sum of the eigenvalues of A, so:

$$7+5+(-3)+(-7)=2=\operatorname{tr}(A)=1+1+1+\text{missing eigenvalue},$$

hence the other eigenvalue of A is -1. We can now conclude that the eigenvalues of A are:

- 1 of multiplicity 3,
- -1 of multiplicity 1.

d) Since the multiplicity of 1 is equal to $\dim E_1$, and the multiplicity of -1 is 1, we conclude that A is diagonalizable.

2. a)

$$\operatorname{rk}(B-I_3)=\operatorname{rk}\begin{pmatrix}1&1&-2\\2&2&-4\\-1&2&-2\end{pmatrix}\underset{\substack{C_2\leftarrow C_2-C_1\\C_3\leftarrow C_3+2C_1}}{=}\operatorname{rk}\begin{pmatrix}1&0&0\\2&0&0\\-1&3&-4\end{pmatrix}=2$$

and

$$\operatorname{rk}(B-2I_3)=\operatorname{rk}\begin{pmatrix}0&1&-2\\2&1&-4\\-1&2&-3\end{pmatrix}\underset{C_1\leftrightarrow C_2}{=}\operatorname{rk}\begin{pmatrix}1&0&-2\\1&2&-4\\2&-1&-3\end{pmatrix}\underset{C_3\leftarrow C_3+2C_1}{=}\operatorname{rk}\begin{pmatrix}1&0&0\\1&2&-2\\2&-1&1\end{pmatrix}=2.$$

b) From the previous question, we know that 1 and 2 are eigenvalues of B. We're missing one eigenvalue, and using the trace trick,

$$4=\operatorname{tr}(B)=1+2+\text{missing eigenvalue}$$

we conclude that 1 is the missing eigenvalue. We hence conclude that the eigenvalues of B are:

- 1 of multiplicity 2;
- 2 of multiplicity 1.

c) By the Rank–Nullity Theorem, $\dim E_1=3-\operatorname{rk}(B-I_3)=1\neq$ multiplicity of 1, and we conclude that B is not diagonalizable.

Exercise 2.

1. Let $n\in\mathbb{Z}$. From the text we know that:

$$w_{n+1}=w_n-\frac{1}{8}w_n+\frac{1}{16}u_n=\frac{7}{8}w_n+\frac{1}{16}u_n$$

and

$$u_{n+1}=u_n+\frac{1}{8}w_n-\frac{1}{16}u_n=\frac{1}{8}w_n+\frac{15}{16}u_n.$$

Hence

$$A=\begin{pmatrix}7/8&1/16\\1/8&15/16\end{pmatrix}.$$

2. By induction:
   - The base case ($n=0$) is obvious since $A^0 = I_2$.
   - Assume that the property holds true for some $n \in \mathbb{N}$. Then:
   $$\begin{pmatrix} w_{n+1} \\ u_{n+1} \end{pmatrix} = A \begin{pmatrix} w_n \\ u_n \end{pmatrix} = AA^n \begin{pmatrix} w_0 \\ u_0 \end{pmatrix} = A^{n+1} \begin{pmatrix} w_0 \\ u_0 \end{pmatrix}$$
   as required.
3. We compute the characteristic polynomial of A:
$$\chi_A(\lambda) = \lambda^2 - \frac{29}{16}\lambda + \frac{13}{16} = (\lambda - 1)\left(\lambda - \frac{13}{16}\right).$$
Hence the eigenvalues of A are 1 and 13/16; since all the eigenvalues of A are of multiplicity 1, we conclude that A is diagonalizable.
We now determine eigenvectors of A:
   - $$E_1\colon \begin{cases} -x/8 + y/16 = 0 \\ x/8 - y/16 = 0 \end{cases} \iff \begin{cases} x = y/2 \\ y = y \end{cases}$$
   hence we choose $X_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
   - $$E_{13/16}\colon \begin{cases} x/16 + y/16 = 0 \\ x/8 + y/8 = 0 \end{cases} \iff \begin{cases} x = -y \\ y = y \end{cases}$$
   hence we choose $X_{13/16} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

We now define:
$$P = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \qquad \text{and} \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & 13/16 \end{pmatrix}.$$
Then P is invertible and:
$$A = PDP^{-1}.$$
Let $n \in \mathbb{N}$. We know that
$$A^n = PD^nP^{-1},$$
and
$$D^n = \begin{pmatrix} 1 & 0 \\ 0 & (13/16)^n \end{pmatrix}.$$
Now
$$P^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix},$$
hence
$$\begin{aligned} A^n &= \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & (13/16)^n \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2(13/16)^n & -(13/16)^n \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} 1 + 2(13/16)^n & 1 - (13/16)^n \\ 2 - 2(13/16)^n & 2 + (13/16)^n \end{pmatrix}. \end{aligned}$$
Finally we conclude:
$$\forall n \in \mathbb{N}, \quad \begin{cases} w_n = \dfrac{1}{3}\left(1 + \left(\dfrac{13}{16}\right)^n\right) w_0 + \dfrac{1}{3}\left(1 - \left(\dfrac{13}{16}\right)^n\right) u_0 \\[2ex] u_n = \dfrac{1}{3}\left(2 - 2\left(\dfrac{13}{16}\right)^n\right) w_0 + \dfrac{1}{3}\left(2 + \left(\dfrac{13}{16}\right)^n\right) u_0. \end{cases}$$

4. We then have:
$$\lim_{n\to+\infty} w_n = \frac{w_0+u_0}{3} \qquad \text{and} \qquad \lim_{n\to+\infty} u_n = \frac{2(w_0+u_0)}{3}.$$
In a distant future from now, we can expect to have twice more unemployed persons than working persons!

Exercise 3.

1. Since $A \in M_3(\mathbb{R})$ has 3 distinct eigenvalues, we conclude that A is diagonalizable.
2. Set
$$P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \qquad \text{and} \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$
Then,
$$A = PDP^{-1}.$$
3. We compute P^{-1}:
$$\begin{cases} x + y = a \\ x + 2y + z = b \\ y + 2z = c \end{cases} \underset{C_2 \leftarrow C_2 - C_1}{\iff} \begin{cases} x + y = a \\ y + z = -a + b \\ y + 2z = c \end{cases}$$
$$\underset{C_3 \leftarrow C_3 - C_2}{\iff} \begin{cases} x + y = a \\ y + z = -a + b \\ z = a - b + c \end{cases} \iff \begin{cases} x = 3a - 2b + c \\ y = -2a + 2b - c \\ z = a - b + c, \end{cases}$$
hence
$$P^{-1} = \begin{pmatrix} 3 & -2 & 1 \\ -2 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix}.$$
4. Hence
$$\begin{aligned} A &= PDP^{-1} \\ &= \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 3 & -2 & -1 \\ -2 & 2 & 1 \\ 1 & -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 0 \\ 1 & 4 & -1 \\ 0 & 2 & -2 \end{pmatrix} \begin{pmatrix} 3 & -2 & 1 \\ -2 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 2 & -1 \\ -6 & 7 & -4 \\ -6 & 6 & -4 \end{pmatrix}. \end{aligned}$$

Exercise 4.

1.
   - $f(u)$:
$$\left[f(u)\right]_{\text{std}} = A \begin{pmatrix} -1 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix} = -[u]_{\text{std}},$$
   hence $f(u) = -u$, hence u is an eigenvector of f associated to the eigenvalue -1.
   - $f(w)$:
$$\left[f(w)\right]_{\text{std}} = A \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix} = 2[w]_{\text{std}},$$
   hence $f(w) = 2w$, hence w is an eigenvector of f associated to the eigenvalue 2.
2. We already have two eigenvalues of f, and using the trace trick we conclude that -1 is the missing eigenvalue. Hence the eigenvalues of f are:
   - -1 of multiplicity 2;

- 2 of multiplicity 1.

We need to determine the dimension of E_{-1}: we'll use the Rank–Nullity Theorem, but we first need the rank of $f+\mathrm{id}$:

$$\begin{aligned}\operatorname{rk}(f+\mathrm{id}) = \operatorname{rk}(A+I_3) = \operatorname{rk}\begin{pmatrix}5 & 0 & 1\\ -8 & -6 & -4\\ 2 & 9 & 4\end{pmatrix} &\underset{C_1\leftrightarrow C_3}{=} \operatorname{rk}\begin{pmatrix}1 & 0 & 5\\ -4 & -6 & -8\\ 4 & 9 & 2\end{pmatrix}\\ &\underset{C_3\leftarrow C_3-5C_1}{=} \operatorname{rk}\begin{pmatrix}1 & 0 & 0\\ -4 & -6 & 12\\ 4 & 9 & -18\end{pmatrix} = 2,\end{aligned}$$

hence $\dim E_{-1} = 3 - \operatorname{rk}(f+\mathrm{id}) = 1 \neq 2$, hence f is not diagonalizable.

3. We have:

$$P = \begin{pmatrix}-1 & 0 & 1\\ -2 & 1 & 0\\ 5 & -1 & -2\end{pmatrix}$$

and we now solve the system associated with P:

$$\begin{aligned}\begin{cases}-x + z = a\\ -2x + y = b\\ 5x - y - 2z = c\end{cases} &\underset{\substack{C_2\leftarrow C_2-2C_1\\ C_3\leftarrow C_3+5C_1}}{\iff} \begin{cases}-x + z = a\\ y - 2z = -2a + b\\ -y + 3z = 5a + c\end{cases}\\ &\underset{C_3\leftarrow C_3+C_2}{\iff} \begin{cases}-x + z = a\\ y - 2z = -2a + b\\ z = 3a + b + c\end{cases}\\ &\iff \begin{cases}x = 2a + b + c\\ y = 4a + 3b + 2c\\ z = 3a + b + c.\end{cases}\end{aligned}$$

Since this system has a unique solution we conclude that P is invertible and that:

$$P^{-1} = \begin{pmatrix}2 & 1 & 1\\ 4 & 3 & 2\\ 3 & 1 & 1\end{pmatrix}.$$

4. We already know the first and last columns of T:

$$T = \begin{pmatrix}-1 & * & 0\\ 0 & * & 0\\ 0 & * & 2\end{pmatrix}.$$

Now,

$$\big[f(v)\big]_{\mathrm{std}} = A\begin{pmatrix}0\\ 1\\ -1\end{pmatrix} = \begin{pmatrix}-1\\ -3\\ 6\end{pmatrix},$$

and by the change of basis formula,

$$\big[f(v)\big]_{\mathscr{B}} = P^{-1}\begin{pmatrix}-1\\ -3\\ 6\end{pmatrix} = \begin{pmatrix}1\\ -1\\ 0\end{pmatrix},$$

hence:

$$T = \begin{pmatrix}-1 & 1 & 0\\ 0 & -1 & 0\\ 0 & 0 & 2\end{pmatrix}.$$

The relation between A, P and T is:

$$A = PTP^{-1}.$$

5. We have:

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad \text{and} \qquad N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

a) Since D is diagonal, its powers are straightforward to obtain:

$$\forall n \in \mathbb{N},\ D^n = \begin{pmatrix} (-1)^n & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & 2^n \end{pmatrix}.$$

Also,

$$N^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{0}_3,$$

hence:

$$\forall n \in \mathbb{N},\ n \geq 2,\ N^2 = \mathbf{0}_3.$$

The other cases are trivial: $N^0 = I_3$ and $N^1 = N$.

b) We use the Binomial Theorem, but we first need to check that D and N commute:

$$DN = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$ND = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

hence $ND = DN$ hence, by the Binomial Theorem, for $n \geq 2$,

$$\begin{aligned} T^n &= (D+N)^n \\ &= \sum_{k=0}^{n} \binom{n}{k} D^{n-k} N^k \\ &= \sum_{k=0}^{1} \binom{n}{k} D^{n-k} N^k \\ &= \binom{n}{0} D^n N^0 + \binom{n}{1} D^{n-1} N^1 \\ &= D^n + nD^{n-1}N \\ &= \begin{pmatrix} (-1)^n & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & 2^n \end{pmatrix} + n \begin{pmatrix} (-1)^{n-1} & 0 & 0 \\ 0 & (-1)^{n-1} & 0 \\ 0 & 0 & 2^{n-1} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} (-1)^n & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & 2^n \end{pmatrix} + n \begin{pmatrix} 0 & (-1)^{n-1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} (-1)^n & n(-1)^{n-1} & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & 2^n \end{pmatrix}. \end{aligned}$$

Note that this formula is also valid for $n = 0$ and $n = 1$.

c) We know that

$$\forall n \in \mathbb{N},\ A^n = PT^nP^{-1},$$

hence

$$A^n = \begin{pmatrix} -1 & 0 & 1 \\ -2 & 1 & 0 \\ 5 & -1 & -2 \end{pmatrix} \begin{pmatrix} (-1)^n & n(-1)^{n-1} & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & 2^n \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 4 & 3 & 2 \\ 3 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 1 \\ -2 & 1 & 0 \\ 5 & -1 & -2 \end{pmatrix} \begin{pmatrix} (2-4n)(-1)^n & (1-3n)(-1)^n & (1-2n)(-1)^n \\ 4(-1)^n & 3(-1)^n & 2(-1)^n \\ 3\cdot 2^n & 2^n & 2^n \end{pmatrix}$$

$$= \begin{pmatrix} (4n-2)(-1)^n + 3\cdot 2^n & (3n-1)(-1)^n + 2^n & (2n-1)(-1)^n + 2^n \\ 8n(-1)^n & (6n+1)(-1)^n & 4n(-1)^n \\ (6-20n)(-1)^n - 6\cdot 2^n & (2-15n)(-1)^n - 2\cdot 2^n & (3-10n)(-1)^n - 2\cdot 2^n \end{pmatrix}$$

Exercise 5. See Figure 4.

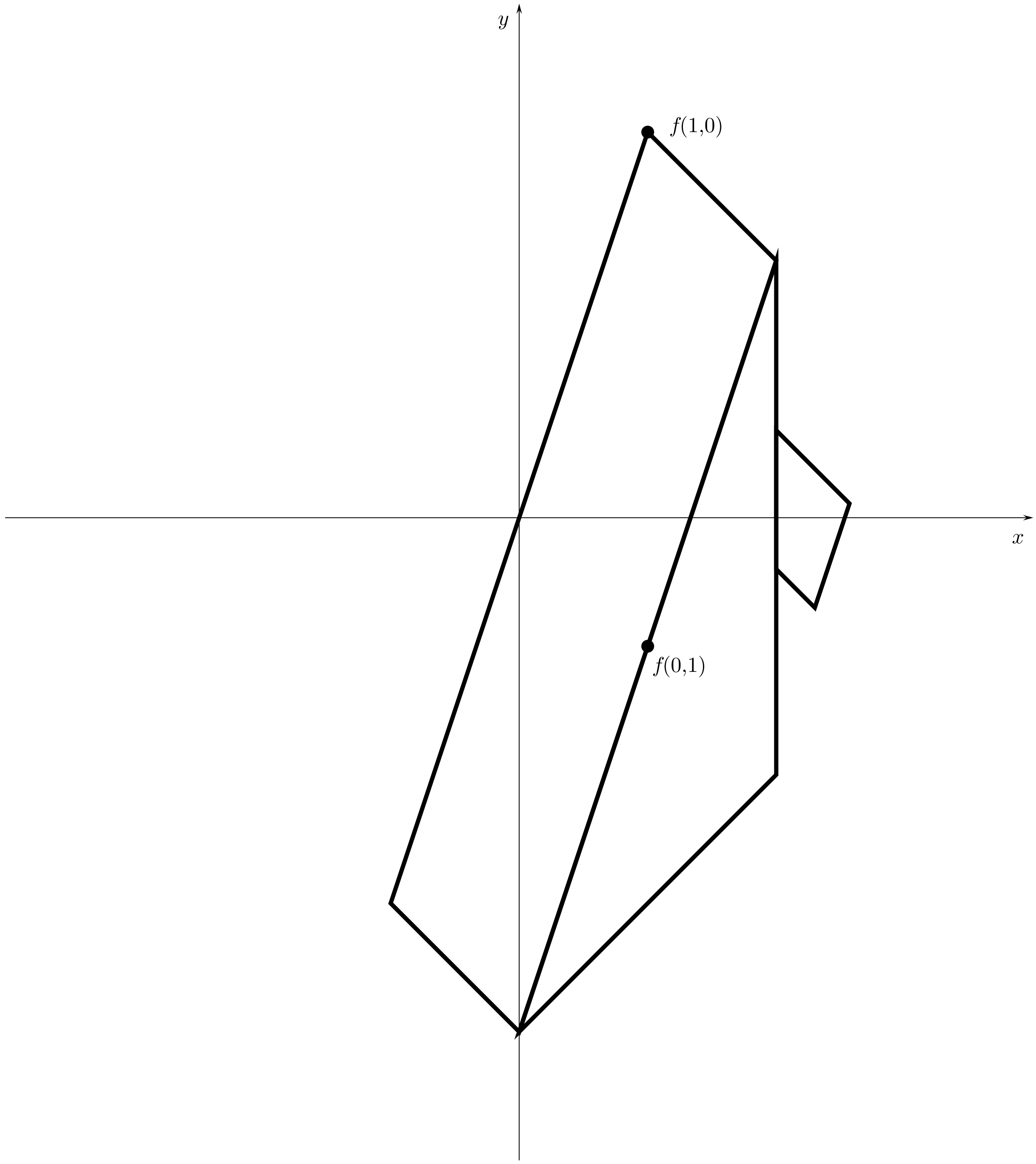

Figure 4 – Image of the house by f.