INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — S2 — Math Test #6 — 2h

June 19, 2019

No documents, no cell phones allowed. Calculators allowed. Cute and fluffy pets allowed (for moral support only).

All your answers must be fully (but concisely) justified, unless noted otherwise.

Exercise 1. The following two questions are independent from each other.

1. Let
$$A = \begin{pmatrix} 7 & 6 & -6 & -6 \\ 4 & 5 & -4 & -4 \\ 4 & 4 & -3 & -4 \\ 8 & 8 & -8 & -7 \end{pmatrix}.$$
   a) What is the rank of $A - I_4$?
   b) Deduce that 1 is an eigenvalue of A, and determine the dimension of the corresponding eigenspace E_1. What can you deduce about the multiplicity of the eigenvalue 1?
   c) Determine all the eigenvalues of A and their multiplicities.
   d) Is the matrix A diagonalizable?

2. Let
$$B = \begin{pmatrix} 2 & 1 & -2 \\ 2 & 3 & -4 \\ -1 & 2 & -1 \end{pmatrix}$$
   a) Determine the rank of $B - I_3$ and of $B - 2I_3$.
   b) Determine all the eigenvalues of B and their multiplicities.
   c) Is the matrix B diagonalizable?

Exercise 2. Assume that the employment situation in some country evolves as follows: among all the people that are unemployed in year n, $1/16$ of them finds a job in year $n + 1$; and among all the people that are employed in year n, $1/8$ of them loses their jobs in year $n + 1$.
We want to know how the employment situation will look like over several years from now.
Given a year $n \in \mathbb{Z}$, define w_n to be the number of working people in year n, and define u_n to be the number of unemployed people in year n.

1. Show that we can model the problem with a matrix $A \in M_2(\mathbb{R})$ that you will determine, such that:
$$\forall n \in \mathbb{Z}, \begin{pmatrix} w_{n+1} \\ u_{n+1} \end{pmatrix} = A \begin{pmatrix} w_n \\ u_n \end{pmatrix}.$$
2. Show that:
$$\forall n \in \mathbb{N}, \begin{pmatrix} w_n \\ u_n \end{pmatrix} = A^n \begin{pmatrix} w_0 \\ u_0 \end{pmatrix}.$$
3. Show that A is diagonalizable and diagonalize A (you must specify the change of basis matrix P and the diagonal matrix D).
4. Compute A^n for $n \in \mathbb{N}$ and deduce an expression of w_n and u_n.
5. Explain what the employment situation will look like in a far future from now.

Exercise 3. Let $A \in M_3(\mathbb{R})$ such that:

- $X_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector of A associated with the eigenvalue 1;
- $X_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ is an eigenvector of A associated with the eigenvalue 2;
- $X_{-1} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ is an eigenvector of A associated with the eigenvalue -1.

Find A.

Hint: your road map should be:

1. *Explain why A is diagonalizable.*
2. *Determine a diagonal matrix D and a change of basis matrix P such that $A = PDP^{-1}$.*
3. *Compute P^{-1}.*
4. *With P, P^{-1} and D, determine A.*

Exercise 4. Let

$$A = \begin{pmatrix} 4 & 0 & 1 \\ -8 & -7 & -4 \\ 2 & 9 & 3 \end{pmatrix},$$

and let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the endomorphism of $\mathbb{R}^3$ such that $[f]_{\text{std}} = A$.
Define the vectors

$$u = (-1, -2, 5), \qquad v = (0, 1, -1), \qquad w = (1, 0, -2),$$

and set $\mathscr{B} = (u, v, w)$ and $P = [\mathscr{B}]_{\text{std}}$.

1. Show that u and w are eigenvectors of f, and specify the eigenvalue they are associated to.
2. Determine the eigenvalues of f and their multiplicities. Is f diagonalizable?
3. Show that P is invertible and compute P^{-1}. From this, we know that $\mathscr{B}$ is a basis of $\mathbb{R}^3$.
4. Determine the matrix $T = [f]_{\mathscr{B}}$. What relation exists between A, P and T? *Hint: the matrix T you obtain should be upper triangular, of the form*
$$T = \begin{pmatrix} \alpha & c & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix},$$
for some coefficients α, β, c you will determine.
5. Write $T = D + N$ with:
$$D = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 0 & c & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$
   a) Compute N^n and D^k for $n \in \mathbb{N}$.
   b) Deduce an explicit expression of T^n for $n \in \mathbb{N}$.
   c) Deduce an explicit expression of A^n for $n \in \mathbb{N}$.

Exercise 5. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the endomorphism of $\mathbb{R}^2$ such that $f(1,0)$ and $f(0,1)$ are shown in Figure 2. Plot on Figure 2 the image by f of the house shown in Figure 1.

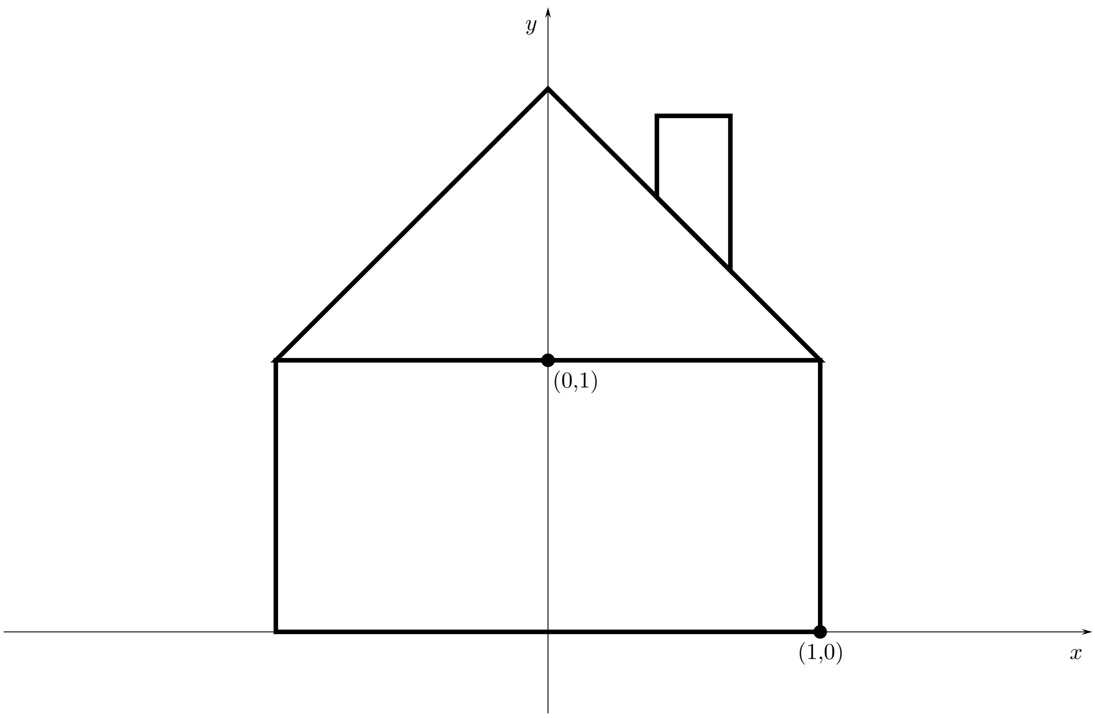

Figure 1 – Original house

Name:

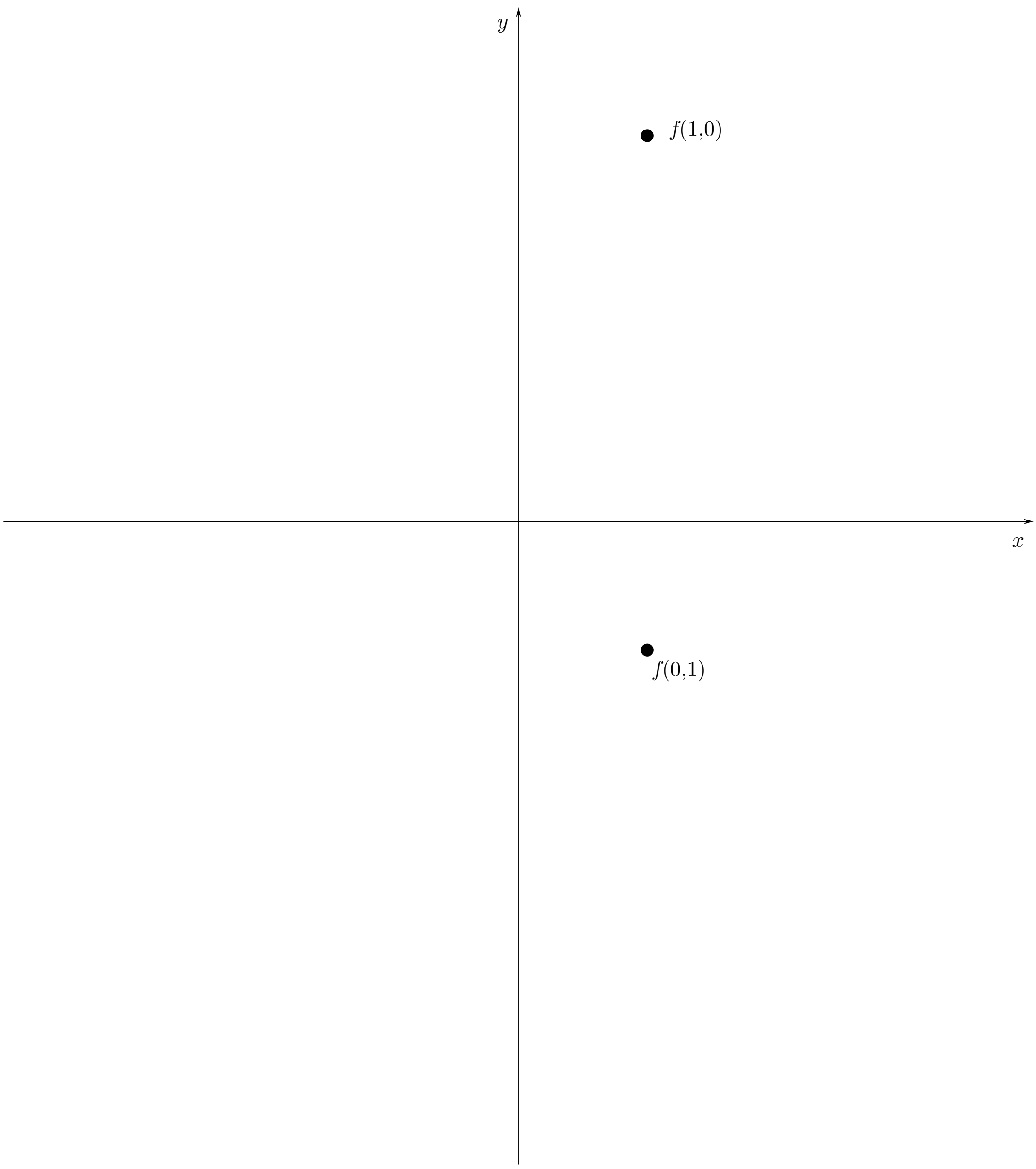

Figure 2 – Plot the image by f of the house of Figure 1

This sheet must be handed in with your test!