# Exam n° 1 – Solutions

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- No documents, no calculators, no cell phones or electronic devices allowed.
- Take a deep breath before starting (everything is going to be ok!) and read entirely the exam before starting. [1]
- All exercises are independent, you can do them in the order that you'd like.
- Please start an exercise at the top of a page (for readability).
- Number single pages, or simply the booklets (*copies doubles*) if multiple : for example 1/3, 2/3, 3/3
- All your answers must be fully (but concisely) justified, unless noted otherwise.
- Redaction and presentation matter! For instance, write full sentences and make sure your '$x$' and '$n$' can be distinguished.
- Questions marked (*) are considered more challenging.
- **Respecting all of the above is part of the exam grade (0.5 points).** Provided rubric is indicative (changes may occur).
- This exam is out 18.5 points. The additional 1.5 points come from your URCHIN quiz grade.

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## Warm-up exercises (9 points)

You are expected to provide some steps for those exercises. Little partial credit will be given for just writing the answer.

**Exercise 1. True or False** Justify briefly why the statement is True or False. If the statement is false, provide the correct answer.

1. The negation of the implication $B \Rightarrow A$ is $\neg A \Rightarrow B$.
2. The negation of $(\forall x \in E, \exists y \in E | f(x) + f(y) \geq 2)$ is $(\exists x \in E, \exists y \in E | f(x) + f(y) < 2)$.
3. The set of solutions of the inequation $\sqrt{2x+3} \geq x$ is $[-1, 3]$.
4. Consider the sets A,B,E with $A, B \subset E$. Then $E \setminus (A \cup B) = (E \setminus A) \cup (E \setminus B)$.
5. For all $a > 0$, $x \mapsto a^x$ and $x \mapsto x^a$ have the same monotonicity over $\mathbb{R}_*^+$.

*Solution.*

1. False : $\neg(B \Rightarrow A) \iff B \wedge \neg A$
2. False : $\neg(\forall x \in E, \exists y \in E | f(x) + f(y) \geq 2) \iff (\exists x \in E | \forall y \in E,\ f(x) + f(y) < 2)$
3. False : let's take a counter-example. For $x = -\dfrac{3}{2}$ we have $\sqrt{2x+3} = 0 \geq -\dfrac{3}{2}$. So the set of solutions is larger than $[-1, 1]$.
   More generally let's find the actual set of solutions. Assuming $x \geq 0$ (then $\sqrt{2x+3}$ is defined) then we can write $\sqrt{2x+3} \geq x \iff 2x + 3 \geq x^2$ which leads in the end to $x \in [0, 3]$. Assuming $-\dfrac{3}{2} \leq x \leq 0$, then $\sqrt{2x+3}$ is defined and the inequation remains satisfied. The overall set of solutions is then $x \in [-\dfrac{3}{2}, 3]$.
4. False : the De Morgan's law is $E \setminus (A \cup B) = (E \setminus A) \cap (E \setminus B)$.

1. Draw a jack-o'-lantern next to your name on the first page once this is done.

5. False : let's take a counter-example. For $a = \frac{1}{2}$, $x \mapsto (\frac{1}{2})^x = e^{-x\ln(2)}$ is decreasing over $\mathbb{R}^+$, and $x \mapsto x^{\frac{1}{2}}$ is increasing over $\mathbb{R}^+$.
More generally, $x \mapsto a^x = e^{x\ln(a)}$ is decreasing over $\mathbb{R}^+$ for $0 < a < 1$ and increasing for $a > 1$, while $x \mapsto x^a = e^{a\ln(x)}$ is increasing over $\mathbb{R}^+$, for all $a > 0$.

□

**Exercise 2.**
Let $n \in \mathbb{N}^*$.

1. Show that for all $k = 1, 2, \ldots, n$, $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$
2. Deduce that $\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$.
3. Show by induction that $\forall n \geq 1$, $\sum_{k=0}^{n} k^3 = \left(\sum_{k=0}^{n} k\right)^2$ (Hint : recall the value of $\sum_{k=0}^{n} k$).

*Solution.* 1. By direct proof : $\forall k = 1, 2, \ldots, n$,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n}{k}\frac{(n-1)!}{(k-1)!((n-1)-(k-1))!} = \frac{n}{k}\binom{n-1}{k-1}$$

2. Using previous question (and operating a change of index) we have

$$\sum_{k=0}^{n} k\binom{n}{k} = \sum_{k=1}^{n} k\binom{n}{k} = n\sum_{k=1}^{n}\binom{n-1}{k-1} = n\sum_{k=0}^{n-1}\binom{n-1}{k} = n(1+1)^{n-1} = n2^{n-1}$$

We recognized Newton's formula $(a+b)^{n-1}$ for $a = b = 1$.

3. First we recall that the arithmetic sum gives us

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2} \Longrightarrow \left(\sum_{k=0}^{n} k\right)^2 = \frac{n^2(n+1)^2}{4}$$

We want to prove by induction the following property

$$P(n) \equiv \sum_{k=0}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

is true for all $n \geq 1$.

— Base case : for $n = 1$ we have $\sum_{k=0}^{1} k^3 = 1$ and $\frac{n^2(n+1)^2}{4} = 1$.

— Induction step : assuming $P(n)$ is true for $n \geq 1$. Let us show that $P(n+1)$ is true.

$$\sum_{k=0}^{n+1} k^3 = \sum_{k=0}^{n} k^3 + (n+1)^3 = \frac{n^2(n+1)^2}{4} + (n+1)^3 = \frac{(n+1)^2(n^2+4n+4)}{4} = \frac{(n+1)^2(n+2)^2}{4}$$

— We conclude that the property is true for all $n \geq 1$.

□

**Exercise 3.**

1. Solve the equation $\sin(3x) = \sin(2x)$ in $\mathbb{R}$.
2. Using addition and duplication formula, show that $\sin(3x) = \sin(x)\left(4\cos^2(x) - 1\right)$.

3. (*) Using previous questions, find the value of $\cos(\frac{\pi}{5})$.

*Solution.* 1. $\sin(3x) = \sin(2x) \iff \begin{cases} 3x = 2x + 2\pi k \\ 3x = \pi - 2x + 2\pi k \end{cases}, k \in \mathbb{Z}$

so the solutions are $\{\pi + 2\pi\mathbb{Z}\} \cup \{\frac{\pi}{5} + \frac{2\pi}{5}\mathbb{Z}\}$.

2. 

$$\sin(3x) = \sin(2x)\cos(x) + \cos(2x)\sin(x) = 2\sin(x)\cos^2(x) + (2\cos^2(x) - 1)\sin(x) = \sin(x)\left(4\cos^2(x) - 1\right)$$

3. Plugging the above expression in the equation we obtain

$$\sin(x)\left(4\cos^2(x) - 1\right) = \sin(2x) = 2\sin(x)\cos(x)$$

Assuming $x \notin \{\pi + 2\pi\mathbb{Z}\}$ we obtain a quadratic polynomial in $X = \cos(x)$ of the form

$$4X^2 - 2X - 1 = 0$$

which admits as roots $X = \frac{1 \pm \sqrt{5}}{4}$. Then we select the one corresponding to $\frac{\pi}{5}$ (positive value) : $\cos(\frac{\pi}{5}) = \frac{1 + \sqrt{5}}{4}$

□

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# Logic (3.5 points)

**Exercise 4.**

Let $n \in \mathbb{N}^*$, and let $(n+1)$ real numbers $x_0, x_1, \ldots, x_n \in [0, 1]$ such that $0 \leq x_0 \leq x_1 \leq \cdots \leq x_n \leq 1$. We want to show that the following proposition

$P \equiv$There exists at least 2 successive numbers out of these real numbers such that their distance is less than $\frac{1}{n}$

is true.

1. Rewrite $P$ using mathematical language, with quantifiers and values $x_i - x_{i-1}$ for instance.
2. Write the negation of $P$.
3. Show $P$ using a proof by contradiction. Provide steps.

*Solution.* 1. $P \equiv \exists i \in [\![1, n]\!] \mid |x_i - x_{i-1}| \leq \frac{1}{n}$.

2. $\neg P \equiv \forall i \in [\![1, n]\!], |x_i - x_{i-1}| > \frac{1}{n}$.
3. Assume $P$ false. Then for all $i = 1, \ldots, n$

$$|x_i - x_{i-1}| > \frac{1}{n} \Longrightarrow \sum_{i=1}^{n} |x_i - x_{i-1}| > 1$$

Since the numbers are ordered we have $|x_i - x_{i-1}| = x_i - x_{i-1}$ and we have a telescopic sum :

$$\sum_{i=1}^{n} |x_i - x_{i-1}| = x_n - x_0 > 1$$

which is impossible since all numbers belong to $[0, 1]$. Therefore $P$ is true.

□

## Functions (5,5 points)

**Exercise 5.**

We consider the function $f : E \to F$ defined by $f(x) = \cosh(\ln(x))$.

1. Provide the domain of definition $E$ of $f$, and the co-domain $F$ corresponding to the direct image of the domain.
2. For $x \in E$, simplify the expression of $f$.
3. Can the function be extended by continuity at 0 ? Justify your answer.
4. **Injectivity :** Consider $f_1 : E_1 \to F_1,\ f_2 : E_2 \to E_1$ two injective functions.
   (a) Recall the definition of $f_1$ being injective.
   (b) Show that $f_1 \circ f_2$ is injective. Provide steps.
   (c) Assume $f_1,\ f_2$ are monotonic (over their respective domains). What can you say about the monotonicity of $f_1 \circ f_2$ ?
   (d) Provide a restriction $\tilde{f}$ of $f$ so that it is injective. Justify your answer (if multiple choices, select the least restricted option).
5. Is $\tilde{f}$ surjective ? What do you conclude ?
6. What can you say about the min, max, inf, sup of $\tilde{f}$ over its domain ?
7. (*) Provide an expression of the inverse (if defined).

*Solution.*

1. $f : \mathbb{R}_*^+ \to [1, +\infty)$.
2. $\forall x \in \mathbb{R}_*^+,\ f(x) = \dfrac{e^{\ln(x)} + e^{-\ln(x)}}{2} = \dfrac{x^2+1}{2x}$.
3. $\lim\limits_{x \to 0^+} \dfrac{x^2+1}{2x} = +\infty$ so the function cannot be extended by continuity at 0.
4. (a) $\forall x, x' \in E_1,\ f_1(x) = f_1(x') \Longrightarrow x = x'$.
   (b) $f_1 \circ f_2 : E_2 \to F_1$. Let $x, x' \in E_2$ such that $(f_1 \circ f_2)(x) = (f_1 \circ f_2)(x')$. Let us show that $x = x'$.
   $(f_1 \circ f_2)(x) = (f_1 \circ f_2)(x') \Longleftrightarrow f_1(f_2(x)) = f_1(f_2(x'))$. Because $f_1$ is injective this implies that $f_2(x) = f_2(x')$ which by injectivity of $f_2$ implies $x = x'$. So the composed function is injective.
   (c) $f_1 \circ f_2$ is also monotonic (increasing if $f_1,\ f_2$ have the same variations, decreasing if they have opposite variations).
   (d) $x \mapsto \ln(x)$ are injective over $\mathbb{R}_*^+$ (strictly monotonic), $x \mapsto \cosh(x)$ is injective over $\mathbb{R}^-$ or $\mathbb{R}^+$. So $f|_{(0,1]}$ and $f|_{[1,+\infty)}$ is injective. We define $\tilde{f} = f|_{[1,+\infty)} : [1, +\infty) \to [1, +\infty)$.
5. Since $\tilde{f}([1, +\infty)) = [1, +\infty)$ it is surjective, therefore bijective.
6. $\min\limits_{[1,+\infty)} (\tilde{f}) = \min[1, +\infty) = 1 = \inf([1, +\infty))$, and there is no max, no sup.
7. $\tilde{f}^{-1}(x) = e^{\text{argch}\,(x)}$.

□