Exam nº 2 – 1 hour 30 minutes

- No documents, no calculators, no cell phones or electronic devices allowed.
- Take a deep breath before starting (everything is going to be ok!) and read entirely the exam before starting.^{0}
- All exercises are independent, you can do them in the order that you'd like.
- Please start an exercise at the top of a page (for readability).
- Number single pages, or simply the booklets (*copies doubles*) if multiple : for example 1/3, 2/3, 3/3
- All your answers must be fully (but concisely) justified, unless noted otherwise.
- Redaction and presentation matter! For instance, write full sentences and make sure your 'x' and 'n' can be distinguished.
- **Respecting all of the above is part of the exam grade (0.5 points).** Provided rubric is indicative (changes may occur).

Warm-up exercises (8 points)

You are expected to provide some steps for those exercises. Little partial credit will be given for just writing the answer.

Exercise 1. True or False Justify briefly why the statement is True or False. If the statement is false, provide the correct answer (if applicable).

1. Given a polynomial $P \in \mathbb{R}_3[X]$, if 2 is root of P then $(X-2)$ divides P'.
2. By composition we have $\lim\limits_{x\to 0^+} 2\arctan(\ln(1-\sin(x))) = 0$.
3. Solving $\cos(x) = \sin(2x)$ over $\mathbb{R}$ leads to the set of solutions $S = \{\frac{\pi}{6}, \frac{\pi}{2}\} + \pi\mathbb{Z}$.

Solution.

1. False : Let's take for example $P(X) = (X-1)(X+1)X$. Then $P(X) \in \mathbb{R}_3[X]$ and 1 is root of P. $P'(X) = X(X+1) + (X-1)(2X+1)$ and $P'(1) = 2 \neq 0$
2. True : $\lim\limits_{x\to 0^+} 2\arctan(\ln(1-\sin(x))) = 2\arctan(0) = 0$.
3. False : $\cos(x) = \sin(2x) \iff \cos(x)(1-2\sin(x)) = 0 \iff \begin{cases}\cos(x) = 0\\ \sin(x) = \dfrac{1}{2}\end{cases} \iff \begin{cases} x = \frac{\pi}{2} + k\pi,\ k\in\mathbb{Z}\\ x = \frac{\pi}{6} + 2k\pi,\ k\in\mathbb{Z}\\ x = \frac{5\pi}{6} + 2k\pi,\ k\in\mathbb{Z}\end{cases}$

   so $S = \{\frac{\pi}{2} + \pi\mathbb{Z}\} \cup \{\frac{\pi}{6} + 2\pi\mathbb{Z}\} \cup \{\frac{5\pi}{6} + 2\pi\mathbb{Z}\}$

□

Exercise 2. Compute **ONE** of the following limits (your choice) :

$$\text{Choice A : } \lim_{x\to 0^+} \frac{e^{\cos(x)} - e}{\sin(x)}, \qquad \text{or} \qquad \text{Choice B : } \lim_{x\to+\infty} \left(1 + \sin\left(\frac{1}{x}\right)\right)^{2x}.$$

If you decide to do both and if one is incorrect, we will only consider the incorrect one (so choose, and choose wisely).

Solution. —

Choice A Using TSEs we have $\cos(x) \underset{0}{=} 1 - \frac{x^2}{2} + o(x^2)$ and $\lim\limits_{x\to 0}\cos(x) = 1$.

By composition we need the $TSE(1)$ of exp : $e^x \underset{1}{=} e + e(x-1) + o(x-1) \iff e^x - e \underset{1}{=} e(x-1) + o(x-1)$. We obtain for the numerator : $e^{\cos(x)} - e \underset{0}{=} -\frac{ex^2}{2} + o(x^2)$.

We obtain for the denominator : $\sin(x) \underset{0}{=} x + o(x)$, therefore $\dfrac{e^{\cos(x)} - e}{\sin(x)} = -\dfrac{x}{2} + o(x) \xrightarrow[x\to 0]{} 0$.

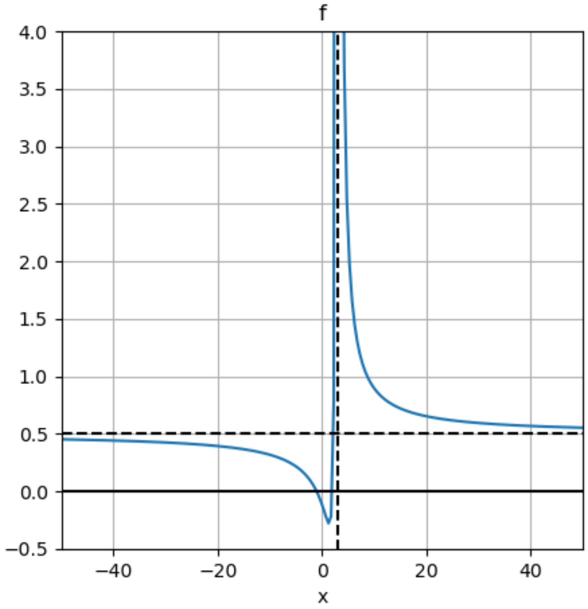
Choice B We have $\left(1 + \sin\left(\dfrac{1}{x}\right)\right)^{2x} = e^{2x\ln(1+\sin\left(\frac{1}{x}\right))}$ $(1 + \sin\left(\dfrac{1}{x}\right) > 0)$. Using TSEs (or asymptotic expansions) we obtain : $\sin\left(\dfrac{1}{x}\right) \underset{+\infty}{=} \dfrac{1}{x} + o\left(\dfrac{1}{x}\right)$ and $\ln(1+X) \underset{0}{=} X + o(X)$ so by composition $2x\ln(1 + \sin\left(\dfrac{1}{x}\right)) \underset{+\infty}{=} 2 + o(1)$ leading to $\lim\limits_{x\to+\infty}\left(1 + \sin\left(\dfrac{1}{x}\right)\right)^{2x} = e^2$.

$\square$

Exercise 3. Consider $f : A \to \mathbb{R}$ such that $f(x) = \dfrac{x^2 - x - 2}{18 + 2x^2 - 12x}$. Provide the domain of definition A. Sketch the graph of f (justify limit behaviors). Be as precise as possible in your answers.

Solution.

1. We can rewrite f as $f(x) = \dfrac{(x+1)(x-2)}{2(x-3)^2}$ so $A = \mathbb{R} \setminus \{3\}$, f has two zeros $(2, -1)$ of multiplicity one, and one pole (3) of multiplicity 2.
2. We find that $\lim\limits_{x\to\pm\infty} f(x) = \dfrac{1}{2}$ (two horizontal asymptotes)
3. We find that $\lim\limits_{x\to 3^\pm} f(x) = +\infty$ (one vertical asymptote at $x = 3$)
4. $f(0) = -\dfrac{1}{9} < 0$



$\square$

Linear systems (5 points)

We expect **full details** on the steps, with proper operations and redaction, and a proper solution written in the end. If details not provided, we will not check your calculations.

Exercise 4. We consider the function $f : \mathbb{R}^4 \to \mathbb{R}^4$ defined by

$$f(x, y, z, t) = (x - t, 2x + y - 3t, 3x + 2y + z - 6t, 4x + 2y + 2z - 8t)$$

1. Compute $\ker(f)$.
2. What conditions do we have on the parameters a, b, c, d so that (a, b, c, d) admits (at least) a pre-image by f ?
3. Is f injective ? surjective ? Justify your answers.
4. Consider the system

$$(S) \begin{cases} 3x + 2y + z - 6t = 1 \\ 4x + 2y + 2z - 8t = 1 \\ x - t = 1 \\ 2x + y - 3t = 1 \end{cases}$$

Does (S) have a (unique) solution ? Justify your answer.

Solution. 1. $\ker(f) = \{(x, y, z, t) \in \mathbb{R}^4 | \, f(x, y, z, t) = 0_{\mathbb{R}^4}\}$. Let's solve the following system and proceed by Gaussian elimination :

$$f(x, y, z, t) = 0_{\mathbb{R}^4} \Leftrightarrow \begin{array}{c} \begin{array}{ccccc} x & y & z & t & b \end{array} \\ \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 2 & 1 & 0 & -3 & 0 \\ 3 & 2 & 1 & -6 & 0 \\ 4 & 2 & 2 & -8 & 0 \end{array}\right) \end{array}$$

$$\begin{array}{c} L_2 \leftarrow L_2 - 2L_1 \\ L_3 \leftarrow L_3 - 3L_1 \\ L_4 \leftarrow L_4 - 4L_1 \\ \Leftrightarrow \end{array} \begin{array}{c} \begin{array}{ccccc} x & y & z & t & b \end{array} \\ \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 2 & 1 & -3 & 0 \\ 0 & 2 & 2 & -4 & 0 \end{array}\right) \end{array} \begin{array}{c} L_3 \leftarrow L_3 - 2L_2 \\ L_4 \leftarrow L_4 - 2L_2 \\ \Leftrightarrow \end{array} \begin{array}{c} \begin{array}{ccccc} x & y & z & t & b \end{array} \\ \left(\begin{array}{cccc|c} 1 & 0 & 0 & 11 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & -2 & 0 \end{array}\right) \end{array}$$

$L4$ and $L3$ are proportional we conclude there are 3 equations for 4 unknowns, so 1 free parameter. We find

$$\ker(f) = \{(x, y, z, t) \in \mathbb{R}^4 | \begin{cases} x = \alpha \\ y = \alpha \\ z = \alpha \\ t = \alpha \end{cases}, \alpha \in \mathbb{R}\} = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}\right)$$

2. Redoing the same steps as before, we solve

$$f(x, y, z, t) = (a, b, c, d) \Leftrightarrow \begin{array}{c} \begin{array}{ccccc} x & y & z & t & \end{array} \\ \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & a \\ 2 & 1 & 0 & -3 & b \\ 3 & 2 & 1 & -6 & c \\ 4 & 2 & 2 & -8 & d \end{array}\right) \end{array}$$

$$\begin{array}{l} L_2 \leftarrow L_2 - 2L_1 \\ L_3 \leftarrow L_3 - 3L_1 \\ L_4 \leftarrow L_4 - 4L_1 \\ \Leftrightarrow \end{array} \begin{array}{c} \begin{array}{cccc} x & y & z & t \end{array} \\ \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & a \\ 0 & 1 & 0 & -1 & b-2a \\ 0 & 2 & 1 & -3 & c-3a \\ 0 & 2 & 2 & -4 & d-4a \end{array}\right) \end{array} \begin{array}{l} L_3 \leftarrow L_3 - 2L_2 \\ L_4 \leftarrow L_4 - 2L_2 \\ \Leftrightarrow \end{array} \begin{array}{c} \begin{array}{cccc} x & y & z & t \end{array} \\ \left(\begin{array}{cccc|c} 1 & 0 & 0 & 11 & a \\ 0 & 1 & 0 & -1 & b-2a \\ 0 & 0 & 1 & -1 & c+a-2b \\ 0 & 0 & 2 & -2 & d-2b \end{array}\right) \end{array}$$

$$\begin{array}{l} L_4 \leftarrow L_4 - 2L_3 \\ \Leftrightarrow \end{array} \quad \begin{array}{c} \begin{array}{cccc} x & y & z & t \end{array} \\ \left(\begin{array}{cccc|c} 1 & 0 & 0 & 11 & a \\ 0 & 1 & 0 & -1 & b-2a \\ 0 & 0 & 1 & -1 & c+a-2b \\ 0 & 0 & 0 & 0 & d+2b-2c-2a \end{array}\right) \end{array}$$

We end up with the compatibility condition $d + 2b - 2c - 2a = 0$.

3. $\ker(f) \neq \{O_{\mathbb{R}^4}\}$ so it is not injective. Question 2 gave us that, in order to find preimages of $\mathbb{R}^4$ we need to consider an hyperplane (but not $\mathbb{R}^4$) so f is not surjective.
4. $(1, 1, 1, 1)$ doesn't satisfy the equation $d + 2b - 2c - 2a = 0$ so S has no solution.

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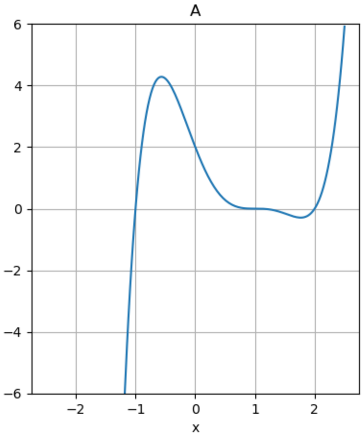
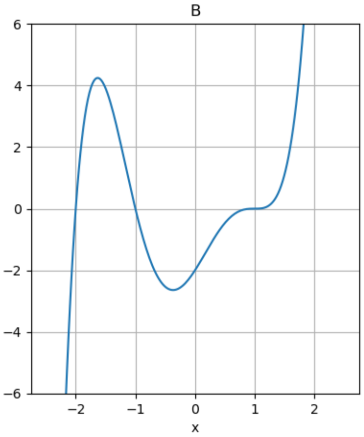
Polynomials (4,5 points)

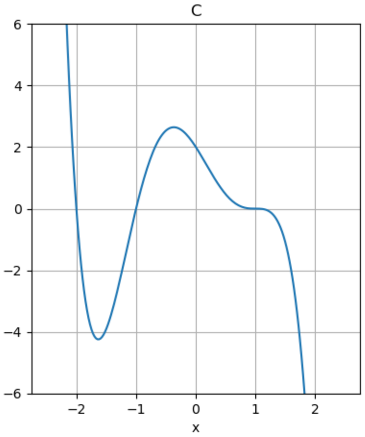
Exercise 5.

Let $a, b \in \mathbb{R}$, we consider the polynomial P defined by $P(X) = X^5 - 4X^3 + 2X^2 + aX + b$.

1. Find a, b such that 1 is root of P or multiplicity at least 2.
2. With the obtained choice of a, b, determine the multiplicity of the root 1.
3. Write the Taylor's formula for P at 1. We expect details on the coefficients computation.
4. Factorize P in $\mathbb{R}$.
5. Given the following graphs, which one corresponds to P? Justify your answer.







Solution. 1. If 1 is root of multiplicity at least 2 then $P(1) = 0$ and $P'(1) = 0$, which leads to the system

$$\begin{cases} 1 - 4 + 2 + a + b = 0 \\ 5 - 12 + 4 + a = 0 \end{cases} \iff \begin{cases} b = -2 \\ a = 3 \end{cases}$$

2. $$P''(X) = 20X^3 - 24X + 4 \Longrightarrow P''(1) = 0$$
$$P'''(X) = 60X^2 - 24 \Longrightarrow P'''(1) = 36 \neq 0$$
so 1 is root of multiplicity 3 and we can write $P(X) = (X-1)^3Q(X)$ with $deg(Q(X)) = 2$.
3. The Taylor's formula gives us
$$P(X) = P(1)+P'(1)(X-1)+P''(1)\frac{(X-1)^2}{2!}+P'''(1)\frac{(X-1)^3}{3!}+P^{(4)}(1)\frac{(X-1)^4}{4!}+P^{(5)}(1)\frac{(X-1)^5}{5!}$$
we have
$$P^{(4)}(X) = 120X \Longrightarrow P^{(4)}(1) = 120$$
$$P^{(5)}(X) = 120 \Longrightarrow P^{(4)}(1) = 120$$

Then using previous question we obtain
$$P(X) = 6(X-1)^3 + 5(X-1)^4 + (X-1)^5$$
4. We can use the previous expression and factorize $(X-1)^3$:
$$P(X) = (X-1)^3(2 + 3X + X^2) = (X-1)^3(X+2)(X+1)$$
where the last factorization can be obtained classically (quadratic equation).
5. P has 3 roots : $-2, -1, 1$ (this eliminates A), $\lim\limits_{x \to +\infty} P(x) = +\infty$, $\lim\limits_{x \to -\infty} P(x) = -\infty$ (this eliminates C) so the associated graph is B.

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Vector Subspace (3 points)

Exercise 6.
Consider the sets
$$F = \{(x,y,z) \in \mathbb{R}^3,\ x + y - 2z = 0\}, \quad G = \{(x,y,z) \in \mathbb{R}^3,\ x = 0\}.$$

1. Show that F, G are vector subspaces.
2. Write $F \cap G$. Is it a vector subspace ?
3. Provide a generating family of F, a generating family of G, and a generating family of $F \cap G$.
4. Is each of those families linearly independent ? What do you conclude ?

Solution. 1. Several options, let's present the classical one. $0_{\mathbb{R}^3} \in F$, let $u = (x_1, y_1, z_1) \in F$, $v = (x_2, y_2, z_2) \in F$, $\alpha, \beta \in \mathbb{R}$. Let us show that $\alpha u + \beta v \in F$.
$$(\alpha x1 + \beta x_2) + (\alpha y_1 + \beta y_2) - 2(\alpha z_1 + \beta z2) = \alpha(x_1 + y_1 - 2_z1) + \beta(x_2 + y_2 - 2z_2) = 0 + 0$$
by definition so F is a vector subspace.
$0_{\mathbb{R}^3} \in G$, let $u = (x_1, y_1, z_1) \in F$, $v = (x_2, y_2, z_2) \in G$, $\alpha, \beta \in \mathbb{R}$. Let us show that $\alpha u + \beta v \in G$.
$$(\alpha x1 + \beta x_2) = 0 + 0$$
by definition so G is a vector subspace.

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0. Draw a snowman next to your name on the first page once this is done.

2. $F \cap G = \{(x, y, z) \in \mathbb{R}^3,\ x = 0, , y - 2z = 0\}$, which is also a vector subspace (the intersection of 2 VS is a VS but not the opposite).

3. We can write $F = \text{Span}\left(\begin{pmatrix}1\\0\\\frac{1}{2}\end{pmatrix}, \begin{pmatrix}0\\1\\\frac{1}{2}\end{pmatrix}\right)$, $G = \text{Span}\left(\begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right)$ and $F \cap G = \text{Span}\left(\begin{pmatrix}0\\1\\\frac{1}{2}\end{pmatrix}\right)$

   The family $\left(\begin{pmatrix}1\\0\\\frac{1}{2}\end{pmatrix}, \begin{pmatrix}0\\1\\\frac{1}{2}\end{pmatrix}\right)$ is generating F by definition.

   The family $\left(\begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right)$ is generating G by definition.

   The family $\left(\begin{pmatrix}0\\1\\\frac{1}{2}\end{pmatrix}\right)$ is generating $F \cap G$ by definition.

4. All those families are linearly independent so they are basis of F, G, $F \cap G$, respectively.

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