0,5 5) (a) $dS = r\,d\theta\, dz$ $\checkmark$ $\overset{dz}{\leftrightarrow}$ $r\,d\theta$ $r = R$!

1 (b) $0 \leq z \leq L$ $\quad 0 \leq \theta \leq 2\pi$

$\frac{dv}{dr} = -Ka\,2r = -\frac{2rK}{R^2}$ — useless

Hence $d\vec{F_{visq}} = \left(\eta \frac{dv}{dr} dS\right)\vec{e_z} = \left(\eta(-Ka\,2r)\,r\,d\theta\, dz\right)\vec{e_z}$

since $r=R$ on the S surface $= -\eta\, Ka\, 2R^2\, d\theta\, dz\, \vec{e_z}$

$$\vec{F_{visq}} = \int_{z=0}^{L}\int_{\theta=0}^{2\pi} -\eta\, Ka\, 2R^2\, d\theta\, dz\, \vec{e_z}$$

Applying Fubini's $= -\eta K_a 2R^2 [\theta]_0^{2\pi} [z]_0^{L} = -\eta\, Ka\, 4R^2 \pi L$ $\checkmark$ $\vec{e_z}$!

exercise 3

1 1) (a) $d\varphi = \nabla\varphi \cdot d\vec{OM}$ 😍 👏 $\Rightarrow$ nice!

$$d\varphi = \frac{\partial\varphi}{\partial r}dr + \frac{\partial\varphi}{\partial\theta}d\theta + \frac{\partial\varphi}{\partial z}dz$$

$$d\vec{OM} = dr\,\vec{e_r} + r\,d\theta\,\vec{e_\theta} + dz\,\vec{e_z}$$

Hence we have $\begin{cases} \frac{\partial\varphi}{\partial r} = \nabla\varphi \cdot d\vec{OM} & \text{along } \vec{e_r} \\ \frac{\partial\varphi}{\partial\theta} = \nabla\varphi \cdot d\vec{OM} & \text{along } \vec{e_\theta} \\ \frac{\partial\varphi}{\partial z} = \nabla\varphi \cdot d\vec{OM} & \text{along } \vec{e_z} \end{cases}$ $\checkmark$

$\Leftrightarrow \begin{cases} \frac{\partial\varphi}{\partial r}dr = \frac{\partial\varphi}{\partial r}\vec{e_r} \cdot dr\,\vec{e_r} \\ \frac{\partial\varphi}{\partial\theta}d\theta = \frac{1}{r}\frac{\partial\varphi}{\partial r}\vec{e_\theta} \cdot r\,d\theta\,\vec{e_\theta} \\ \frac{\partial\varphi}{\partial z}dz = \frac{\partial\varphi}{\partial z}\vec{e_z} \cdot dz\,\vec{e_z} \end{cases}$ By identification we have $\nabla\varphi = \frac{\partial\varphi}{\partial r}\vec{e_r} + \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\vec{e_\theta} + \frac{\partial\varphi}{\partial z}\vec{e_z}$ $\checkmark$

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exercise 1 :

1 1) $\vec{V} = \vec{rot}(\vec{U}) = \begin{pmatrix} \frac{\partial U_z}{\partial y} - \frac{\partial U_y}{\partial z} \\ \frac{\partial U_x}{\partial z} - \frac{\partial U_z}{\partial x} \\ \frac{\partial U_y}{\partial x} - \frac{\partial U_x}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2x - x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix}$ $\checkmark$

1 2) $\vec{U}$ derive from a potential if df is exact [such that $df = \vec{U} \cdot d\vec{OM}$.]

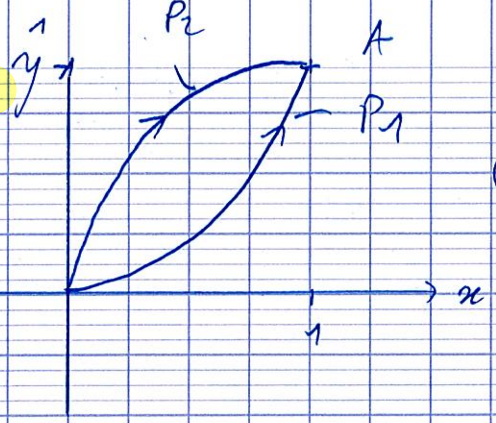
$\frac{\partial U_x}{\partial y} = x \neq \frac{\partial U_y}{\partial y} = 2x$ $\checkmark$

Hence df is not exact hence $\vec{U}$ does not derive from a potential. $\checkmark$

1 3) The rotational vector field's flux V is equal (since it is a double integral) $\checkmark$ to the circulation of the vector field U since it is a simple integral. $\checkmark$ $\checkmark$
The right-side corresponds to a flux $\checkmark$ and the left-side to a circulation.

4)(a)

0,5 a)



$\odot\ \vec{e_z}$

1,5 (b) $\mathcal{C}_{P_1} = \int_{P_1} \vec{U}\cdot d\vec{OM} = \int_{P_1} \begin{pmatrix} t^3 \\ t^2 \\ 0\end{pmatrix}\begin{pmatrix} dt\\ 2t\,dt \\ 0\end{pmatrix} = \int_{t=0}^{1} 3t^3\, dt$

$\begin{cases} x = t \in [0,1] \\ y = t^2 \\ z=0\end{cases}$ $= \left[\frac34 t^4\right]_0^1 = \frac34$

$\mathcal{C}_{P_2} = \int_{P_2} \vec U \cdot d\vec{OM} = \int_{t=0}^1 \begin{pmatrix} t\sqrt t \\ t^2 \\ 0\end{pmatrix}\begin{pmatrix} dt \\ \frac{1}{2\sqrt t}dt \\ 0\end{pmatrix}$ $\begin{cases} x = t \in[0,1] \\ y=\sqrt t \\ z = 0\end{cases}$

$= \int_0^1 \left[t^{\frac32} + \frac12 t^{\frac32}\right] dt = \frac32 \left[\frac25 t^{\frac52}\right]_0^1$

$= \frac35$

0,5 5) (a) $\varphi_\Sigma = \iint_\Sigma \vec U \cdot d\vec S$ $\quad d\vec S = dx\,dy\ \vec{e_z}$

$= \iint_\Sigma \begin{pmatrix} xy \\ x^2 \\ 0\end{pmatrix}\cdot\begin{pmatrix}0\\0\\dx\,dy\end{pmatrix} = 0$

1 (b) i) $\varphi'_\Sigma = \iint_\Sigma \vec V \cdot d\vec S = \iint_\Sigma \begin{pmatrix}0\\0\\x\end{pmatrix}\cdot\begin{pmatrix}0\\0\\dx\,dy\end{pmatrix}$

$= \int_{x=0}^{1}\int_{y=x^2}^{\sqrt x} x\,dy\,dx = \int_0^1 x(\sqrt x - x^2)\,dx$

$= \left[\frac25 x^{\frac52} - \frac14 x^4\right]_0^1$

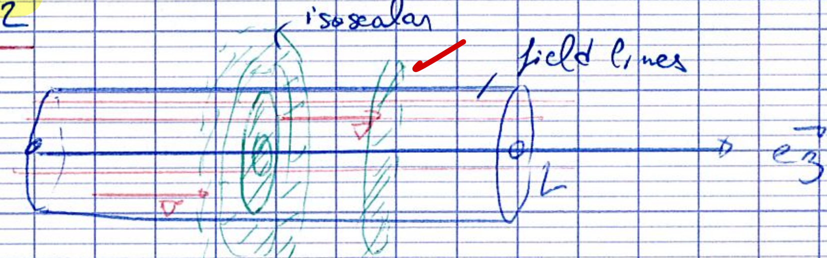
$= \frac25 - \frac14 = \frac{3}{20}$

0,5 ii) $\mathcal{C}_T = \mathcal{C}_{P_1} - \mathcal{C}_{P_2} = \frac34 - \frac35 = \frac{3}{20} = \varphi'_\Sigma(\vec V)$

so the GR formula holds true for this example.

exercise 2

0,5 1)



1 2) We know that $\vec v = v_z(r,z)\,\vec{e_z}$

Hence $v_\theta = v_r = 0$

Hence $\frac1r \frac{\partial(v_r r)}{\partial r} + \frac1r\frac{\partial v_\theta}{\partial\theta} + \frac{\partial v_z}{\partial z} = \frac{\partial v_z}{\partial z} = 0$

If $\vec v = v_z(r,z)$ then $\frac{\partial v_z}{\partial z} \neq 0$ so it wouldn't respect the fact the fluid is incompressible hence $\vec v = v_z(r)\,\vec{e_z}$

0,5 3) $\vec v = K(1-ar^2)\,\vec{e_z}$

When $r=R$, $\vec v = \vec 0 \iff K(1-aR^2)=0$

$\iff a = \frac{1}{R^2}$

1,5 4) Since the velocity only depends on r which is a positive constant, (thus) $\vec v$ is oriented along $+\vec{e_z}$.

$d\vec S = d\vec D = r\,dr\,d\theta\ \vec{e_z}$

Hence $Q = \int_{r=0}^{R}\int_{\theta=0}^{2\pi} \overset{\vec v}{\begin{pmatrix}0\\0\\K(1-\frac{1}{R^2}r^2)\end{pmatrix}}\begin{pmatrix}0\\0\\r\,dr\,d\theta\end{pmatrix}$

Applying Fubini's theorem: $Q = K\int_{r=0}^{R} \left(1-\frac{1}{R^2}r^2\right) r\,dr \int_{\theta=0}^{2\pi} d\theta = 2\pi K\left[\frac12 r^2 - \frac{1}{4R^2} R^4\right]_0^R$

$= \frac24 \pi K R^2 = \frac12 \pi K R^2$

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1,5 b) $\varphi(r,\theta) = \frac{\cos\theta}{r^\alpha} + \beta$

$$\frac{\partial \varphi}{\partial r} = \cancel{\frac{\cos\theta}{-\alpha+1} r^{-\alpha+1}} = -\alpha r^{-\alpha-1}\cos\theta$$

$$\frac{\partial \varphi}{\partial \theta} = -\frac{1}{r^\alpha}\sin\theta$$

$$\begin{cases} \frac{\partial \varphi}{\partial r} = -\frac{2\cos\theta}{r^3} \\ \frac{1}{r}\frac{\partial \varphi}{\partial \theta} = \frac{-\sin\theta}{r^3} \end{cases} \Leftrightarrow \begin{cases} \cancel{-}\alpha r^{-\alpha-1}\cdot\cancel{\cos\theta} = \cancel{-}\frac{2\cancel{\cos\theta}}{r^3} \\ \cancel{-}\frac{\cancel{\sin\theta}}{r^{\alpha+1}} = \cancel{-}\frac{\cancel{\sin\theta}}{r^3} \end{cases}$$

$$\Leftrightarrow \begin{cases} \alpha r^{-\alpha-1} = 2r^{-3} \\ r^{-\alpha-1} = r^{-3} \end{cases} \Leftrightarrow \begin{cases} \alpha = 2 \\ \alpha = 2 \end{cases}$$

$\varphi(r,\theta) = \frac{\cos\theta}{r^\alpha} + \beta$ when $r \to +\infty$ $\varphi = 0$

$\Leftrightarrow \beta = 0$

1 2) $\varphi(r,\theta) = k \Leftrightarrow \frac{\cos\theta}{r^2} = k$

$\Leftrightarrow \frac{1}{r^2} = \frac{k}{\cos\theta} \Leftrightarrow r = \sqrt{\frac{\cos\theta}{k}}$

When $\theta = \frac{\pi}{2}$, $r = 0$ hence the solid / red curves corresponds to the equipotentials.

The field lines are orthogonal to the equipotentials since $\vec{E}$ derive from a scalar potential dif.

Hence they do correspond to the dotted lines.

(2) 3) Trigonometric direction where $x \geq 0$, $y \geq 0$

Hence $0 \leq \theta \leq \frac{\pi}{2}$.

Since $x^2 + y^2 = 1 \iff r = 1$ ~~$0 \leq r \leq 1$~~

Hence $\mathcal{E}_r = \int_{\theta=0}^{\frac{\pi}{2}} \begin{pmatrix} \frac{2\cos\theta}{r^3} \\ \frac{\sin\theta}{r^3} \end{pmatrix} \cdot \begin{pmatrix} dr \\ r\,d\theta \end{pmatrix} = \int_{\theta=0}^{\frac{\pi}{2}} \sin\theta \, d\theta$

$= [-\cos\theta]_0^{\frac{\pi}{2}} = 1$

$\vec{OM} = r\vec{e_r}$

$\frac{d\vec{OM}}{dt} = \frac{dr}{dt}\vec{e_r} + r\frac{d\theta}{dt}\vec{e_\theta}$

exercise 4:

(0,5) 1) F_x corresponds to the scalar field along $\vec{e_x}$ of the vector field $\vec{F}$

F_y corresponds to the scalar field along $\vec{e_y}$ of the vector field $\vec{F}$.

(0,5) 2) It would return an error message because we can't divide by zero

$\frac{2x^2 - y^2}{(x^2+y^2)^{\frac{5}{2}}} \Rightarrow x$ and y must be different from

(0,5) 3) quiver represents the arrow of the vectors on a graph.

So Python will plot the vector field $\vec{F} = \begin{pmatrix} \frac{2x^2 - y^2}{(x^2+y^2)^{\frac{5}{2}}} \\ \frac{3xy}{(x^2+y^2)^{\frac{5}{2}}} \end{pmatrix}$ on a grid in 2D. Knowing that $0.5 \leq x \leq 2.5$, $0.5 \leq y \leq 2.5$ with a step of 12.

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