Mobile phones, calculators and all documents are forbidden.
Duration: 2 hours. An indicative grading scheme is given over 20 points.

Exercise 1 ($\sim$ 7 points)

We place ourselves in $(O, \vec{e}_x, \vec{e}_y, \vec{e}_z)$, an orthonormal space frame. Let $\vec{U} = U_x\vec{e}_x + U_y\vec{e}_y + U_z\vec{e}_z$ be a vector field of $\mathbb{R}^3$. We define the vector $\overrightarrow{\text{rot}}(\vec{U})$ by:

$$\overrightarrow{\text{rot}}(\vec{U}) = \nabla \times \vec{U} = \left(\frac{\partial U_z}{\partial y} - \frac{\partial U_y}{\partial z}\right)\vec{e}_x + \left(\frac{\partial U_x}{\partial z} - \frac{\partial U_z}{\partial x}\right)\vec{e}_y + \left(\frac{\partial U_y}{\partial x} - \frac{\partial U_x}{\partial y}\right)\vec{e}_z$$

The Green-Riemann formula (in the US it's just called Green's theorem), denoted (GR), applied onto a vector field $\vec{U}$ of class $\mathcal{C}^1$ over a surface (Σ) of $\mathbb{R}^2$, which is bounded by a closed curve Γ of class $\mathcal{C}^1$ piecewise, oriented in the trigonometric direction:

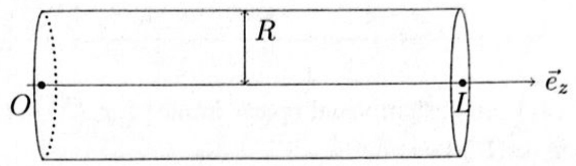
$$(GR) : \int_\Gamma U_x dx + U_y dy = \iint_\Sigma \left(\frac{\partial U_y}{\partial x} - \frac{\partial U_x}{\partial y}\right) dxdy$$

By the way, Green's theorem is a specific case of Stokes' theorem, seen during the last practical. In the remainder of this exercise, we will consider the plane vector field

$$\vec{U} = (xy)\vec{e}_x + (x^2)\vec{e}_y$$

1. Calculate the expression for the vector field $\vec{V} = \overrightarrow{\text{rot}}(\vec{U})$.
2. Does $\vec{U}$ derives from a potential? Explain your answer.
3. Give an interpretation for both sides of the (GR) formula. Which is a circulation, which is a flux? Justify your answer.
4. Consider the arc of the parabola $\mathcal{P}_1$ of equation $y = x^2$, for $x \in [0, 1]$, and the arc of the parabola $\mathcal{P}_2$ of equation $x = y^2$, for $y \in [0, 1]$, each of these arcs oriented as if starting from O. Let A be the point of coordinates $(1, 1)$.
   (a) Plot these two parabola arcs on the same graph. Draw an arrow on each arc to indicate the orientation.
   (b) Calculate the circulation of $\vec{U}$ on $\mathcal{P}_1$ then of $\mathcal{P}_2$.
5. The two arcs of the parabola define a closed surface with interior (Σ). The orientation of (Σ) is given by the vector $\vec{e}_z$.
   (a) What is the flux of $\vec{U}$ through (Σ)?
   (b) i. Calculate the flux of $\vec{V}$ through (Σ).
       ii. Does the (GR) formula hold true for this example?

Exercise 2 (Fluid flow in a pipe ∼ 5,5 points)
Space is referred to an orthonormal reference frame $(O, \vec{e}_x, \vec{e}_y, \vec{e}_z)$. We are interested in the flow of a fluid of viscosity η in a horizontal cylindrical pipe, of axis (Oz), radius R and length L. This flow is caused by the thrust of a piston upstream of the pipe. The result is a pressure difference ΔP (called head loss) between the two ends of the pipe. The pressure field P in the pipe is a function of z only: $P = P(z) > 0$.



The problem will be treated in cylindrical coordinates. Thus, the expression of the velocity field associated with the flow of this fluid would be of the form: $\vec{v} = v_r\vec{e}_r + v_\theta\vec{e}_\theta + v_z\vec{e}_z$.

In the particular case of the flow studied, it can be expressed simply as

$$\vec{v} = v_z(r, z)\vec{e}_z.$$

1. Without any calculations, draw a diagram showing the $\vec{v}$ field lines and the isoscalar (isobaric) pressure field surfaces.

2. The fluid is incompressible. The mathematical translation of this characteristic is given by the following relationship:
$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$$
From this expression, show that the v_z component of the velocity field is independent of z.

3. Speed is given by $\vec{v} = K(1 - ar^2)\vec{e}_z$ with K and a strictly positive constants. Knowing that the velocity is zero on the cylinder wall, determine the expression for the constant a.

4. The flux of $\vec{v}$ through D, which is a straight section of the cylinder, i.e. a disk centered along (Oz), of radius R, parallel to the xOy plane, is called *volumetric flow*. Calculate the flow rate Q (the flux) as a function of R and K, knowing that the domain D is oriented in the direction of the velocity.

5. We will now consider the viscous forces (due to the velocity field in the pipe) on the lateral surface S_ℓ of the cylinder (shaded in the figure). We give the expression for the viscous force experienced by a surface element dS of normal $\vec{e}_r$ (recall that $v = |\vec{v}| = K(1 - ar^2)$):
$$\overrightarrow{dF_{visq}} = \left(\eta\frac{dv}{dr}dS\right)\vec{e}_z$$
   (a) Recall the expression for dS, an elementary surface located on the cylinder's lateral surface.
   (b) The resultant of the viscous forces acting on the entire lateral surface is obtained by
$$\overrightarrow{F_{visq}} = \iint_{S_\ell} \overrightarrow{dF_{visq}}.$$
   Calculate $\overrightarrow{F_{visq}}$ and express it as a function of η, R, L, a and K.

Exercise 3 ($\sim$ 5,5 points)
Consider the vector field $\vec{E} = E_r\vec{e}_r + E_\theta\vec{e}_\theta$ (created by an electrostatic dipole), defined in the local polar reference frame $(O, \vec{e}_r, \vec{e}_\theta)$ by the following components:

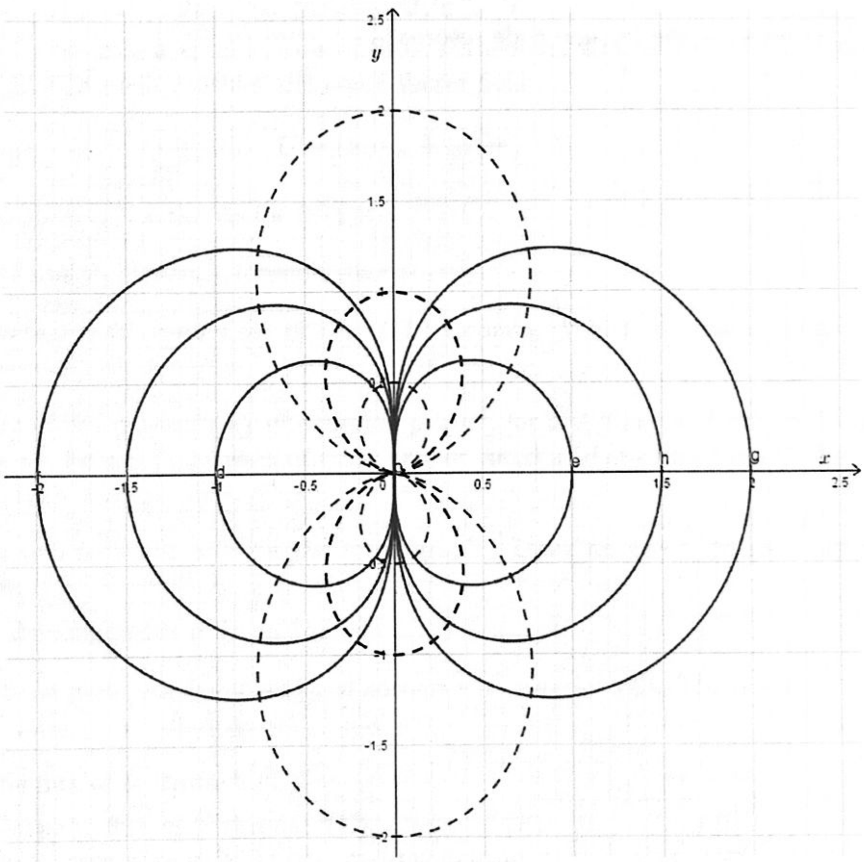
$$E_r = \frac{2\cos(\theta)}{r^3} \text{ et } E_\theta = \frac{\sin(\theta)}{r^3}$$

1. (a) From the definition of the gradient, find the expression of the gradient vector in the local polar basis:

$$\overrightarrow{\text{grad}}\phi = \nabla\phi = \frac{\partial\phi}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\vec{e}_\theta.$$

   (b) Let $\phi(r,\theta) = \dfrac{\cos(\theta)}{r^\alpha} + \beta$, where $\alpha \in \mathbb{N}^*$ and $\beta \in \mathbb{R}$. Determine α and β such that $\vec{E} = -\overrightarrow{\text{grad}}\,\phi$ with $\phi = 0$ when $r \longrightarrow \infty$ (very far away from the dipole).

2. Below are some curves, some dashed and some solid. Without complicated calculations, can you recognize which are equipotentials of ϕ? Knowing the equipotentials, can you deduce which are field lines of $\vec{E}$? (Justify briefly, no need for calculations).



3. Calculate the circulation of $\vec{E}$ along the quarter-circle defined by

$$\begin{cases} x \geq 0 \\ y \geq 0 \\ x^2 + y^2 = 1 \end{cases},$$

oriented in the trigonometric direction. (Hint: convert the path to polar coordinates, i.e., values for r and θ on which the path is defined)

Exercise 4 (∼ 2 points)
Consider the following Python code:

```
1  import numpy as np
2  import matplotlib.pyplot as plt
3  def F_x(x,y):
4      return (2*x**2-y**2)/(x**2+y**2)**(5/2)
5
6  def F_y(x,y):
7      return (3*x*y)/(x**2+y**2)**(5/2)
8
9  x = np.linspace(0.5, 2.5, 12)
10 y = np.linspace(0.5, 2.5, 12)
11 X, Y = np.meshgrid(x, y)
12 plt.quiver(X,Y,F_x(X,Y),F_y(X,Y))
```

1. What do the functions `F_x` and `F_y` represent?
2. What happens if we start the `np.linspace` on lines 9 and 10 at the value 0 instead of 0.5?
3. Can you describe what the Python code returns?