MNTES - S2 Written Exam 1

April 7th, 2023, 16h. Duration: 1 hour and 30 minutes

Guidelines

Not only your results, but especially your ability to clearly justify them and then critically analyze them will be evaluated. You are also reminded to take care in the spelling and presentation of your papers. No documents or calculators are allowed. The scale is given as an indication. **There are three independent problems.**

Exercise 1: Surface of a 2D shape ($\sim$ 4 pts.)

Consider a domain D above the line $y = 1$, and under the curve $y = e^{-x+1}$, for $x \in [0, 1]$.

1. Sketch the domain D.
2. Express the domain D as normal in x, therefore using the format:

$$D = \{(x, y) \in \mathbb{R}^2 \mid a < x < b \ \text{ and } \ \alpha(x) < y < \beta(x)\}$$

   with a, b, α, β to be determined.
3. Calculate the area A of D.
4. What would change if you have to use the other choice? Provide the alternative integral expression of A, with the bounds explicitly (no computation expected).

Exercise 2: Easter chocolate eggs ($\sim$ 7 pts.)

It's almost Easter and time for Easter eggs! While enjoying your delicious chocolate eggs, you may wonder about their mass. Did you know that we can provide an equation for the egg? Let us consider the 2-dimensional equation of the egg provided by Hügelschäffer. The egg is bounded by two circles: one centered at O of radius b, and one centered at $(d, 0)$ of radius a (see figure 1). The shape of the egg is given by the Cartesian equation

$$y^2 = \frac{b^2(a^2 - (x-d)^2)}{a^2 - d^2 + 2dx}.$$

In this problem **we set** $a = 6$ **cm,** $b = 4$ **cm, and** $d = 1$ **cm**, which leads to the egg in figure 1.

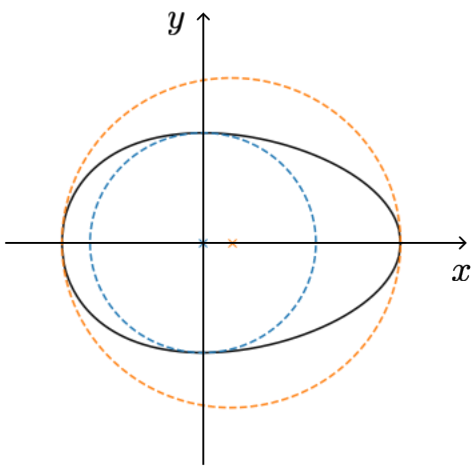


Figure 1: Sketch of the egg (in black). The two dashed circles are used to create the shape of the egg: one centered at O of radius b (blue), and one centered at $(d, 0)$ of radius a (orange).

1. Show that the values x such that $\dfrac{b^2(a^2-(x-d)^2)}{a^2-d^2+2dx} = 0$ are $x_1 = d-a$ and $x_2 = d+a$. Do these values match what is expected from figure 1 above? Explain your reasoning.

2. Determine the expression of the 2-dimensional domain E bounded by the egg's equation, as normal in x, i.e., express it in the format:
$$E = \{(x,y) \in \mathbb{R}^2 \mid \alpha < x < \beta \ \text{ and } \ \gamma(x) < y < \delta(x)\},$$
with α, β, γ, δ to be determined.

3. We consider the surface mass density of the egg $\sigma = \sigma_0\sqrt{35+2x}$ g/cm^2. Express the 2-dimensional mass M_{2D} as a double integral based on your description of E.

4. Rewrite the integrand of M_{2D} in the form of $\sqrt{1-A^2}$ where A is a function of x to be determined.

5. The mass of the 3-dimensional egg is $M_{3D} = 2\pi M_{2D}$. Calculate the 3D mass of the egg. *Hint: use the previous question and perform a change of variables that involves a trigonometric function.*

6. Without calculations, what can you say about the coordinates of the center of mass of the 3D egg ?

Exercise 3: Archimedes' principle ($\sim$ 9 pts.)

We propose to verify the Archimedes' principle (in other words the buoyancy of an object immersed into a fluid) on an example.

We consider a half-ball ($\mathscr{B}$) immersed under water (see figure 2). The surface of this half-ball consists of a half-sphere (Σ) of radius R, of center O, taken as origin of the reference frame, and of a disk ($\mathscr{D}$) (upper face) included in the plane of equation $z = 0$.

We consider the following conventions:

- the constant μ ($kg.m^{-3}$) denotes the volumetric density of water, the constant g ($N.kg^{-1}$) denotes the acceleration of gravity, and P_0 ($N.m^{-2}$) denotes the pressure at the surface of the water
- the Cartesian basis is given by $(O, \vec{e}_x, \vec{e}_y, \vec{e}_z)$, where $\vec{e}_z$ is the vertical ascending
- the spherical coordinates (r, θ, φ) are associated to the local frame $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\varphi)$, and $\overrightarrow{OM} = r\,\vec{e}_r$.
- the cylindrical coordinates (r, θ, z) of axis (Oz) are associated to the local frame $(\vec{e}_r, \vec{e}_\theta, \vec{e}_z)$, and $\overrightarrow{OM} = r\,\vec{e}_r + z\,\vec{e}_z$.

$\boxed{\text{Look closely at Figure 2 with all notations.}}$

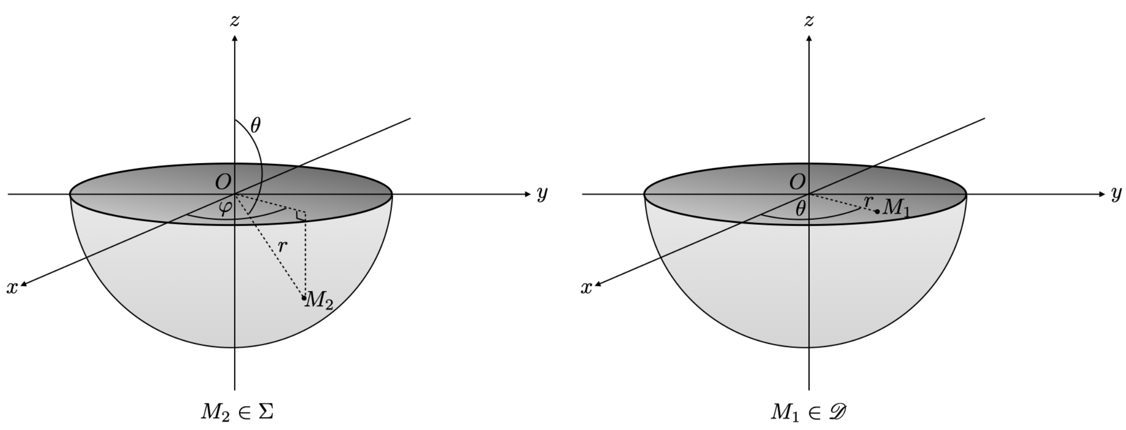


Figure 2: Diagram of the half-ball and associated coordinate systems.

1. **Intervals of coordinates**. Specify the intervals of the following:
   (a) Intervals for the spherical coordinates (r, θ, φ) for points belonging to the half-ball (filled half-sphere) ($\mathscr{B}$),
   (b) Intervals for the cartesian coordinates (x, y, z) for points belonging to the half-ball (filled half-sphere) ($\mathscr{B}$) (several choices are possible),

(c) Intervals for the spherical coordinates (r, θ, φ) for the points on the surface of the half-sphere (Σ),

(d) Intervals for the cylindrical coordinates (r, θ, z) for the points on the surface of the disk $(\mathscr{D})$.

2. **Computation of the mass.** Express the mass of the half-ball, M, as a triple integral (specify your choice of coordinates) then compute it.

The goal is to compute all the forces exerted on the half-ball and to compare them. To that aim we will compute the weight $\vec{P}$, and two contact forces exerted by the water on the immersed half-ball: $\vec{F_1}$ acting on surface (Σ), and $\vec{F_2}$ acting on surface $(\mathscr{D})$.

3. Compute $F_1 = \iint_{\mathscr{D}} P_0 \, dS$, dS being the elementary surface of the disk $\mathscr{D}$. Using question 1, specify your choice of coordinates, write the integral bounds and dS explicitly in that case.

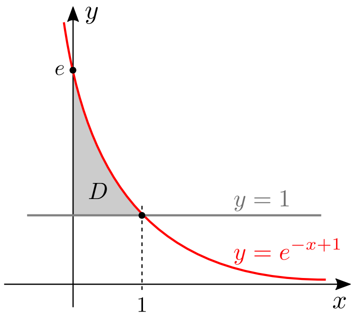
4. Compute $F_2 = \iint_{\Sigma} [P_0 - \mu g z] \cos\theta \, dS$, dS being the elementary surface of the half-sphere Σ. Using question 1, specify your choice of coordinates, write the integral bounds and dS explicitly in that case. **Don't forget to express the function to integrate within the chosen coordinate system.**

5. **Computation of forces.** Using results from previous questions, compute $\vec{P} = -Mg\vec{e}_z$, $\vec{F_1} = -F_1\vec{e}_z$, and $\vec{F_2} = -F_2\vec{e}_z$. Compare $\vec{F_1} + \vec{F_2}$ to $\vec{P}$. Comment on the result.

6. **To go further.** The contact force $\vec{F_2}$ on the half-sphere could have other components. What calculations are needed to show that the other components of $\vec{F_2}$ are nil? You will give details of the literal expressions to be calculated, but you will not perform these calculations.

Solutions and Grading Scheme for IE1S2
SCAN First 2022-2023 - 07/04/2023

Instructions: items in red are graded, items in black are for information only

EX1	**4 pts + 0,5 bonus**
1.1  	**Tot: 1 pt** 0.5 (exp function + correct domain) 0.5 (ref. points)
1.2 $D = \left\{(x, y) \in \mathbb{R} \ \mid \ 0 \leq x \leq 1, \ \ 1 \leq y \leq e^{-x+1}\right\}$	**Tot: 1pt** 0,5 range in x 0,5 range in y
1.3 $A = \iint_D \mathrm{d}S = \int_{x=0}^{1} \int_{y=1}^{e^{-x+1}} \mathrm{d}x\,\mathrm{d}y$ $A = \int_0^1 \left(e^{-x+1} - 1\right) \mathrm{d}x = \left[-e^{-x+1} - x\right]_0^1$ $A = e - 2 \approx 0.7$	**Tot: 1pt** 0,25 0,25 + 0,25 0,25
1.4 Write is as normal in y: $D = \left\{(x, y) \in \mathbb{R} \ \mid \ 0 \leq y \leq e, \ \ 0 \leq x \leq 1 - \ln(y)\right\}$ $A = \iint_D \mathrm{d}S = \int_{y=1}^{e} \int_{x=0}^{1-\ln(y)} \mathrm{d}x\,\mathrm{d}y$ (Bonus) The domain is simple	**Tot: 1pt + 0.5 bonus** 0,5 0,5 (Bonus: 0,5)

EX2	**7 pts**
2.1 $\dfrac{b^2\left(a^2 - (x-d)^2\right)}{a^2 - d^2 + 2xd} = 0$ $\Longrightarrow b^2\left(a^2 - (x-d)^2\right) = 0 \quad \text{and} \quad a^2 - d^2 + 2xd \neq 0$ $x = d \pm a \quad \text{and} \quad x \neq \dfrac{a^2 - d^2}{2d}$ Since the egg is bounded by the yellow circle, the point $x = d \pm a$ matches the intersection of the orange circle with the x − axis	**Tot: 1 pt** 0.5 (for at least the 1st condition) 0.25 (for the solution of the first condition) 0.25 (For the conclusion)

2.2 $E = \left\{ (x,y) \in \mathbb{R} \;\mid\; d-a \le x \le d+a,\; b\sqrt{\dfrac{a^2-(x-d)^2}{a^2-d^2+2xd}} \le y \le -b\sqrt{\dfrac{a^2-(x-d)^2}{a^2-d^2+2xd}} \right\}$ or $E = \left\{ (x,y) \in \mathbb{R} \;\mid\; -5 \le x \le 7,\; 4\sqrt{\dfrac{36-(x-1)^2}{35+2x}} \le y \le -4\sqrt{\dfrac{36-(x-1)^2}{35+2x}} \right\}$	**Tot: 1pt** 0,25 per bound (in either numerical or analytical form)
2.3 $M_{2D} = \iint_E \sigma\, \mathrm{d}S \quad = \quad \int_{x=-5}^{7} \left(\int_{y=-4\sqrt{\frac{36-(x-1)^2}{35+2x}}}^{4\sqrt{\frac{36-(x-1)^2}{35+2x}}} \sqrt{35+2x}\;\; \mathrm{d}y \right) \mathrm{d}x$ $M_{2D} = 8\sigma_0 \int_{-5}^{7} \sqrt{36-(x-1)^2}\;\; \mathrm{d}x$	**Tot: 1pt** 0.5 (correct setup of the integral) 0.25 (if wrong bounds agreeing with previous) 0.5 (for single integral or equivalent form)
2.4 $M_{2D} = 8\sigma_0 \int_{-5}^{7} \sqrt{36-(x-1)^2}\;\; \mathrm{d}x \quad = \quad 8\sigma_0 \int_{-5}^{7} 6\sqrt{1-\left(\dfrac{x-1}{6}\right)^2}\;\; \mathrm{d}x$ with $A = \dfrac{x-1}{6}$	**Tot: 1pt** 0.75 0.25
2.5 Change of variable: $\dfrac{x-1}{6} = \cos(\theta) \implies \mathrm{d}x = -6\sin(\theta)\;\mathrm{d}\theta$ New bounds: $x = -5 \implies \theta = \pi$ and $x = 7 \implies \theta = 0$ (Give all the points if <u>equivalent correct</u> change of variable other than trigonometric) $M_{3D} = 2\pi \times 8\sigma_0 \int_{-5}^{7} 6\sqrt{1-\left(\dfrac{x-1}{6}\right)^2}\;\; \mathrm{d}x \;=\; 96\pi\sigma_0 \int_{\pi}^{0} 6\sqrt{1-\cos^2\theta}\;\; (-6)\sin\theta\, \mathrm{d}\theta$ $= \; 576\pi\sigma_0 \int_0^{\pi} \sin^2\theta\, \mathrm{d}\theta \;=\; 576\pi\sigma_0 \int_0^{\pi} \left(\dfrac{1}{2} - \dfrac{\cos(2\theta)}{2}\right) \mathrm{d}\theta \;=\; 576\pi\sigma_0 \left[\dfrac{\theta}{2} - \dfrac{\sin(2\theta)}{4}\right]_0^{\pi}$ $= 288\pi^2\sigma_0$ grams	**Tot: 2pt** 0,5 (for correct change of variable. Accept as correct if sin() instead of cos()) 0,5 0,75 0,25 (only with units)
2.6 The 3D shape is symmetrical in rotation w.r.t. the $x-$axis (or equiv. formulation) The mass density only depends on x (or equiv. formulation) $\implies$ the center of mass is on the $x-$axis : $G\left(G_x,\, G_y = 0,\, G_z = 0\right)$	**Tot: 1pt** 0.25 0.25 0.5

EX3	**10 pts**
3.1 **a)** $\mathscr{B}(r,\theta,\varphi): \begin{array}{l} 0 \leqslant r \leqslant R \\ \frac{\pi}{2} \leqslant \theta \leqslant \pi \\ 0 \leqslant \varphi \leqslant 2\pi \end{array}$	**Total: 2,5 pt** 0,5
b) $\mathscr{B}(x,y,z): \begin{array}{l} -R \leqslant x \leqslant R \\ -\sqrt{R^2-x^2} \leqslant y \leqslant \sqrt{R^2-x^2} \\ -\sqrt{R^2-x^2-y^2} \leqslant z \leqslant \sqrt{R^2-x^2-y^2} \end{array}$ (accept the two other choices on y and z)	1
c) $\Sigma(r,\theta,\varphi): \begin{array}{l} r = R \\ \frac{\pi}{2} \leqslant \theta \leqslant \pi \\ 0 \leqslant \varphi \leqslant 2\pi \end{array}$	0,5
d) $\mathscr{D}(r,\theta,z): \begin{array}{l} 0 \leqslant r \leqslant R \\ 0 \leqslant \theta \leqslant 2\pi \\ z = 0 \end{array}$	0,5
3.2 $M = \iiint_{\mathscr{B}} \mu \, \mathrm{d}V,$ using spherical (easiest) and taking μ to be a constant (homogeneous):	**Total: 1,5pt** 0,25
$M = \mu \int_{\varphi=0}^{2\pi} \int_{\theta=\pi/2}^{\pi} \int_{r=0}^{R} r^2 \sin(\theta) \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\varphi \quad = \quad \mu \int_0^{2\pi} \mathrm{d}\varphi \cdot \int_{\pi/2}^{\pi} \sin(\theta) \, \mathrm{d}\theta \cdot \int_0^R r^2 \, \mathrm{d}r$	0,5
$= \quad 2\pi\mu \frac{R^3}{3} \left[-\cos\theta\right]_{\pi/2}^{\pi}$	0,5
$= \quad \frac{2R^3}{3}\pi\mu \quad \text{kilograms}$ (Full points to other coordinates based <u>correct</u> solutions. <u>Non-integral based solutions are not accepted</u>)	0,25 (only with units)

3.3	**Total: 1pt**
$F_1 = \iint_{\mathscr{D}} P_0 \, \mathrm{d}S,$	
taking P_0 as a constant and using the polar coordinates determined in 3.1.d) with $\mathrm{d}S = r\,\mathrm{d}\theta\,\mathrm{d}r$ we have:	0,25+0,25
$F_1 = P_0 \int_{\theta=0}^{2\pi} \int_{r=0}^{R} r \, \mathrm{d}r\,\mathrm{d}\theta \quad = \quad P_0 \pi R^2$	0,5
(Full points to other coordinates based <u>correct</u> solutions. <u>Non integral based solutions are not accepted</u>)	
3.4	**Total: 3pt**
$F_2 = \iint_{\Sigma} \left[P_0 - \mu g z\right] \cos\theta \,\mathrm{d}S,$	
using the spherical coordinates determined in 3.1.c) with $\mathrm{d}S = R^2 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$ and $z = R\cos\theta$ we have:	0,25+0,25+0,25
$F_2 = \int_{\varphi=0}^{2\pi} \int_{\theta=\pi/2}^{\pi} \left[P_0 - \mu g z\right] \cos\theta \; R^2 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$	0,25
$= R^2 \int_{\varphi=0}^{2\pi} \int_{\theta=\pi/2}^{\pi} \left[P_0 \cos\theta \sin\theta - \mu g R \cos^2\theta \sin\theta\right] \mathrm{d}\theta\,\mathrm{d}\varphi$	1
$= 2\pi R^2 \left[\frac{1}{2} P_0 \sin^2\theta + \frac{1}{3}\mu g R \cos^3\theta\right]_{\pi/2}^{\pi} \quad = \quad -P_0 \pi R^2 - \frac{2}{3}\pi \mu g R^3$	0,5+0,5
(Full points to other coordinates based <u>correct</u> solutions. <u>Non integral based solutions are not accepted</u>)	
3.5	**Total: 1pt**
$\overrightarrow{P} = -Mg\,\overrightarrow{e}_z = \frac{2}{3}\pi\mu g R^3\,\overrightarrow{e}_z$	0,25
$\overrightarrow{F}_1 + \overrightarrow{F}_2 = -F_1\,\overrightarrow{e}_z - F_2\,\overrightarrow{e}_z = \cancel{-P_0\pi R^2\,\overrightarrow{e}_z} + \cancel{P_0\pi R^2\,\overrightarrow{e}_z} + \frac{2}{3}\pi\mu g R^3\,\overrightarrow{e}_z,$	0,5
hence $\overrightarrow{P} = \overrightarrow{F_1} + \overrightarrow{F_2}$, which means that the contact forces balance out the weight and the object is in equilibrium (or any equivalently intelligent formulation) (No points for conclusion without calculations)	0,25

3.6	Total: 1pt
Since the shape is spherical, the contact force $\overrightarrow{F}_2$ is, in reality, perpendicular to the surface, hence $\overrightarrow{F}_2 = -F_{2r}\overrightarrow{e}_r$, in spherical coordinates and spherical local frame	0,25
Due to the symmetry of rotation around the axis (Oz) we can express $\overrightarrow{F}_2$ in cylindrical coordinates and cylindrical local frame:	
$\overrightarrow{F}_2 = -F'_{2r}\overrightarrow{e}_r - F_{2z}\overrightarrow{e}_z$, with $F'_{2r} \neq F_{2r}$ due to the change in coordinate system from spherical to cylindrical	0,25
When we take the sum of all $\overrightarrow{F}_2$ contributions around the spherical surface Σ we get the sum of the components on the radial direction $\overrightarrow{F}'_{2r}$ and on the vertical direction $\overrightarrow{F}_{2z}$. Since question 3.5 specifies that the F_2 calculated in 3.4 is on $-\overrightarrow{e}_z$, then we have already calculated $\overrightarrow{F}_{2z}$ in 3.4, and $\overrightarrow{F}_{2z} \neq 0$.	0,25
But we notice that for any depth z, every $\overrightarrow{F}'_{2r}(z)$ contribution there's an opposite $\overrightarrow{F}'_{2r}(z)$ contribution of the same amount due to the symmetry of rotation around the axis (Oz). Hence, $\sum \overrightarrow{F}'_{2r}(z) = 0 \ \forall z$, which justifies that $\overrightarrow{F}_2 = \sum \overrightarrow{F}_{2z}$ from question 3.4.	0,25