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Scan First
School Year 2022-2023

MNTES – S1 – End of Semester Written Exam

20th of January, 2022 **Total time: 2h**

Calculators, phones and documents are NOT authorized. Indicative grading scale

Exercise 1 (4 points)

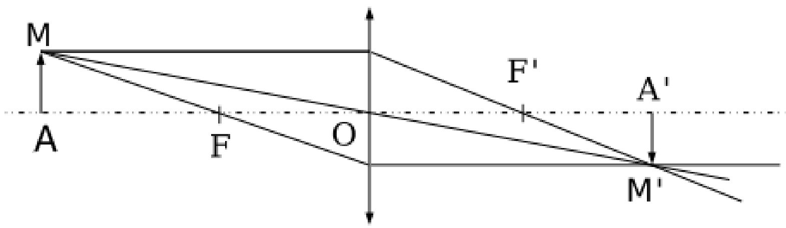
Consider the differential form $\omega = (x^2 + y^2 + 2x)dx + 2ydy$ defined on $\mathbb{R}^2$.

1) Show that ω is not closed.

2) Let $\varphi: \mathbb{R} \to \mathbb{R}$ be a function of class C^1 and $\omega_1 = \varphi(x)\omega$ be a differential form.
   a. Determine a function φ such that ω_1 is closed, with $\varphi(0) = 1$.
   b. With the function φ found previously, is it possible to conclude that ω_1 is exact? Why?
   c. Find the functions $f: \mathbb{R}^2 \to \mathbb{R}$ such that $df = \omega_1$.
   Hint : what is the first derivative of x^2e^x with respect to x ?

Exercise 2 (5 points)

We provide hereafter a ray-diagram, giving the image $A'M'$ of an object AM through a thin lens of focal length $f' = \overline{OF'}$:



The conjugate equation gives the relationship between $a' = \overline{OA'}$, $a = \overline{OA}$ and $f' = \overline{OF'}$:

$$\frac{1}{a'} - \frac{1}{a} = \frac{1}{f'}$$

In this expression, a, a' can be positive or negative. They can be measured and their uncertainties Δa and $\Delta a'$ can be directly estimated during the experiment.

1) Uncertainties:
   a. Give the expression of $f'(a, a')$ for $(a, a') \in \mathbb{R}^2, a \neq a'$.
   b. Express the differential df'.
   c. Deduce the absolute uncertainty $\Delta f'$ on f', as a function of Δa and $\Delta a'$.
   d. Numerical application: Students measured $a = (40.0 \pm 2.0)\ cm$ and $a' = (-40.00 \pm 0.40)\ cm$. Compute f' and $\Delta f'$and write the result as $f' = (\cdots \pm \cdots)\ unit$.

2) Small variations: During a second experiment, the students use a thin lens with a focal length $f' = +100\ mm$. The object AM is initially placed at the position $a = -20\ cm$.
   a. Compute the position a' of the image $A'M'$.

b. The students move the object by a quantity δa. Give the expression of the variation $\delta a'$ of the image position.
c. Numerical application: Compute $\delta a'$ for $\delta a = -1\ cm$.

Exercise 3: movement of a load carried by a crane

The exercise is composed of 3 independent parts.

Part A (3 points)

A load M is rotating at the end of a cable [OM] of length L, attached to the end of a crane boom.
The load will be interpreted as a point M in the direct orthonormal frame $(O, \vec{e}_x, \vec{e}_y, \vec{e}_z)$, whose axes are shown in Figure 1 (in the Appendix).
M has a circular rotation around the vertical z-axis (dotted line in Figure 1 in the Appendix). The point O is the end of the crane's boom. The undirected angle $\widehat{POM}$ is equal to π/4 radians.

1) On figure 1 in the appendix, represent:
   - The spherical coordinates r, θ, φ of point M,
   - The local spherical frame $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\varphi)$ at point M.

2) Give the expression of vector $\overrightarrow{OM}$ using spherical coordinates, first in the Cartesian frame $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$, then in the local frame $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\varphi)$.

3) Propose a parametrization of the trajectory of point M using spherical coordinates.

Part B (5 points)
It is now assumed that under the action of friction, the original circular motion is transformed into a motion of oscillations in a plane φ = constant in spherical coordinates.

To study this new motion, **a new reference frame is chosen and polar coordinates will be used (see Figure 2 in the Appendix)**. θ now denotes the angle of the polar coordinates.

g denotes the gravity of Earth ($g \simeq 9.81 \mathrm{m.s^{-2}}$) and L is constant.

1) On figure 2 in the appendix, define the local polar frame at point M.
2) Give the expression of $\overrightarrow{e_r}$ and $\overrightarrow{e_\theta}$ in the Cartesian frame, and deduce the expression of $\dot{\overrightarrow{e_r}} = \frac{d\overrightarrow{e_r}}{dt}$.
3) a. Give the expression of $\overrightarrow{OM}$ and $\vec{v}$ in the local polar frame.
   b. Express the acceleration $\vec{a} = \frac{d\vec{v}}{dt}$ of point M in the local polar frame.

4) Projecting the second Newton's law onto the direction of $\overrightarrow{e_\theta}$, it is possible to obtain the differential equation of motion, which can be put into the form:
$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$
   a. Show with differential calculus that if θ is very small, then $sin(\theta) \simeq \theta$.

b. Give the general solutions of the differential equation (1) below, with unknown function θ of variable t, in the case where θ is considered very small:

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0 \quad (1)$$

Partie C (3 points)

Because of a failure of the cable, the load M falls. The differential equation verified by the speed $v(t)$ of point M is:

$$m\frac{dv}{dt} + \lambda\, v = mg \quad \text{with } \lambda \in \mathbb{R}_+^* \quad (3)$$

1) Determine the solution $v(t)$ of equation (3), knowing that $v(0) = 0$.

2) Complete the dotted lines of the Python code given in appendix and add the necessary elements in order to have a step h of 0.01 and a table " v " which contains the approximate values of the solution of (4), on the interval [a,b], for this value of h.

APPENDIX TO BE RETURNED	LAST NAME : FIST NAME : GROUP :

Figures 1 and 2, to be completed in Part A and Part B, respectively

Part A

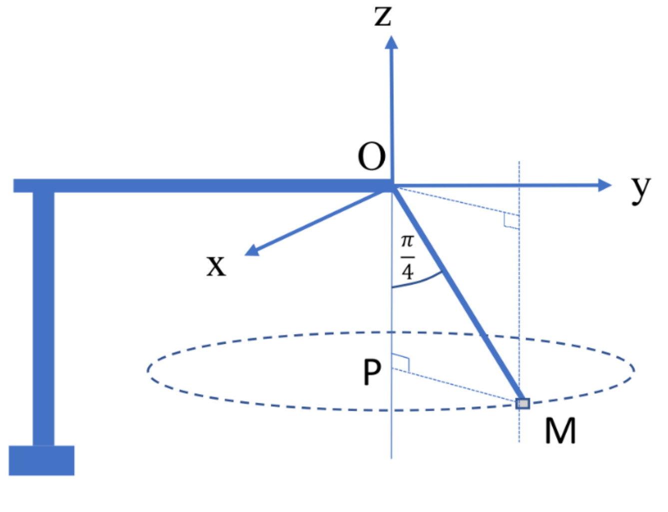


Figure 1: Circular motion of the load M around the z-axis. The trajectory is represented in dotted line.

Part B

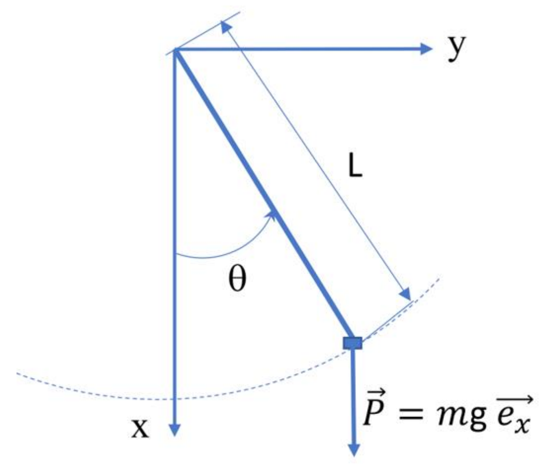


Figure 2: Motion of the load M of mass m. The trajectory is represented by the dotted arc of a circle with radius L, the length of the cable.

Python code, to be completed in Part C

```
Import ........................


a = 0
b = 10
m = 1
mu = 1
..............................

n = ..........               # number of subintervals
t = np.ones(n) * 0           # initialisation of the array t of size n
v = np.ones(n) * 0           # initialisation of v with the initial
condition
h = (b-a)/n                      # definition of the step

for k in range(1,n):
      t[k]=t[k-1]+h
      v[k]=v[k-1]+h*.................................
print(v)
```