1st MNTES exam – Semester 1
November 18th, 2022. Duration: 1.5 h

No document allowed. No mobile phone, no electronic devices (like apple watch). Non-programmable calculator allowed. The proposed grading scale is only indicative.
It is also reminded that the general clarity and cleanness of your paper may also be taken into account.

Exercise 1 (10 points)

An electromagnetic wave is defined by its electric field $\vec{E}$ and its magnetic field $\vec{B}$. Let $\vec{n}$ be the unit vector giving the direction along which the electromagnetic wave propagates.

The vectors are such that: $\vec{n} \cdot \vec{E} = 0$, $\vec{n} \cdot \vec{B} = 0$ and $\vec{B} = \dfrac{1}{V} \vec{n} \times \vec{E}$ where V is the velocity of the wave along the direction of the unit vector $\vec{n}$.

1. (a) Schematically represent $\vec{E}$, $\vec{B}$ and $\vec{n}$ at the same point. You do not need to care about the vector lengths, but you need to explain the directions.
   (b) Give the relation between the length of $\vec{B}$ and that of $\vec{E}$.

In the following, we consider a direct orthonormal basis $(O; \vec{e_x}, \vec{e_y}, \vec{e_z})$. A vector $\vec{k}$, collinear to vector $\vec{n}$ with the same direction, is defined as $\vec{k} = 0.3\vec{e_x} + 0.4\vec{e_y}$. Moreover, in this basis, $\vec{E} = E_z \vec{e_z}$.

2. Give the coordinates of $\vec{n}$.
3. Deduce the coordinates of $\vec{B}$.
4. Give the angle between $\vec{B}$ and (Ox), then the angle between $\vec{B}$ and (Oz) (up to an angle π).
5. Make a scheme with the vectors $\vec{E}$, $\vec{B}$ and $\vec{n}$ correctly placed in the basis $(O; \vec{e_x}, \vec{e_y}, \vec{e_z})$.
6. Determine a Cartesian equation of the plane containing the vectors $\vec{E}$ and $\vec{B}$, and containing the point $A(2, 1, 0)$.
7. The power P of the wave passing through a surface of area S and of normal vector $\vec{u}$ can be computed using the following expression:

$$P = S \left(\vec{E} \times \frac{1}{\mu_0} \vec{B} \right) \cdot \vec{u} \quad \text{where} \quad \mu_0 \quad \text{is a constant}$$

   (a) Find a unit vector $\vec{u}$ in the xy-plane , for which the power P is nil.
   (b) For $\vec{u} = \vec{e_x}$, express P as a function of S, E_z, V and μ_0.

Exercise 2 (5 points + 1 bonus point)

In this exercise, i is the complex number verifying $\mathrm{i}^2 = -1$.

Let z_1, z_2 and z be three complex numbers such that

$$z_1 = \sqrt{2} + \mathrm{i}\sqrt{2} \quad ; \quad z_2 = -2\mathrm{i} \quad ; \quad \frac{1}{z} = \frac{1}{z_1} + \frac{1}{z_2}.$$

1. Write z_1 and z_2 in their exponential forms.

2. Give the Cartesian form of z and highlight its real and its imaginary parts.

3. Deduce the modulus of z (the result should be given as a reduced fraction).

4. Show that $z = \dfrac{\mathrm{e}^{-\mathrm{i}\frac{\pi}{8}}}{\cos\left(\frac{3\pi}{8}\right)}$, with sufficient detail in the calculations.
(Hint: you may factorize the complex number $\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{\mathrm{i}\varphi}$ by $\mathrm{e}^{\mathrm{i}\frac{\theta+\varphi}{2}}$).

5. (bonus) From the previous questions, deduce the exact value of $\cos\left(\frac{3\pi}{8}\right)$.

Exercise 3 (5 points + 1 bonus point)

Using an orthonormal direct basis $(O; \vec{e_x}, \vec{e_y}, \vec{e_z})$, let $\overrightarrow{OM}(t) = x(t)\vec{e_x} + y(t)\vec{e_y} + z(t)\vec{e_z}$ be the position vector of a point $M(t)$, defined as:

$$\begin{cases} x(t) = a\cos(\omega t) \\ y(t) = a\sin(\omega t) \\ z(t) = -bt^2 + ct \\ \quad t \in [0, 2] \end{cases}$$

with $a = 2$ m, $\omega = 2\pi$ rad.s^{-1}, $b = 1$ m.s^{-2} and $c = 2$ m.s^{-1}.

1. Express the cylindrical coordinates of point M as a function of t.

2. Graph the trajectory of M in the plane (xOy), as well as its altitude $z(t)$.

3. Give the Cartesian coordinates of the velocity vector $\vec{v} = \dfrac{\mathrm{d}\overrightarrow{OM}}{\mathrm{d}t}$, then compute its value $\|\vec{v}\|$ at time $t = 0.5$ s.
Knowing that $\vec{e_z}$ is directed upwards, does point M move upwards or downwards? Explain your answer.

4. (bonus) Determine the maximum distance between the origin O of the basis and point $M(t)$ for $t \in [0, 2]$.