Correction of the 2^{nd} MNTES exam – Semester 1
January 19^{th}, 2024

Exercise 1: Differential form (2 points)

1. with $\omega = Pdx + Qdy + Rdz$, we have $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = z^2$ $\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} = 2yz$ $\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} = 2xz + 1$ Hence ω is closed	0.75
2. $\omega = df = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}$ $f(x,y,z) = xyz^2 + yz + z^2 + C \quad \text{with} \quad C \in \mathbb{R}$ $f(0,0,0) = 1 \quad \Longleftrightarrow C = 1$ Therefore $f(x,y,z) = xyz^2 + yz + z^2 + 1$	1 (expression of f with constant) 0.25 (value of the constant)

Exercise 2: Parametrization (2 points)

$\begin{cases} x(t) = cos(t) + 1 \\ y(t) = 2sin(t) + 2 \\ t \in [0, 2\pi)] \end{cases}$ (accept all other consistent parametric descriptions)	0.5 $(sin(t), cos(t))$ 0.5 (2 in front of $sin(t)$) 0.5 (center) 0.5 (range t)

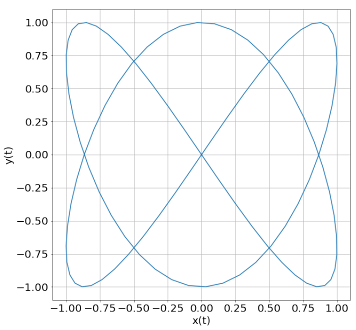
Exercise 3 - Kinematics (6 points)

1. $\begin{cases} r(t) = 2 \\ \theta(t) = 2t \quad \text{as} \quad \dot{\theta} = 2 \quad \text{and} \quad \theta(0) = 0 \\ z(t) = 4cos(4t) \\ t \in [0; \pi] \end{cases}$	0.5
2. $\overrightarrow{OM} = r\vec{e_r} + z\vec{e_z} = 2\vec{e_r} + 4cos(4t)\vec{e_z}$ $\overrightarrow{OM} = rcos\theta\vec{e_x} + rsin\theta\vec{e_y} + z\vec{e_z} = 2cos(2t)\vec{e_x} + 2sin(2t)\vec{e_y} + 4cos(4t)\vec{e_z}$	0.5 0.5
3. $\vec{v} = \frac{d\overrightarrow{OM}}{dt} = r\dot{\theta}\vec{e_\theta} + \dot{z}\vec{e_z} = 4\vec{e_\theta} - 16sin(4t)\vec{e_z}$ $\vec{v} = -4sin(2t)\vec{e_x} + 4cos(2t)\vec{e_y} - 16sin(4t)\vec{e_z}$ $v = \|\|\vec{v}\|\| = 4\sqrt{1 + 16sin^2(4t)}$	0.5 0.5 0.5
4. $\vec{v}$ is horizontal when $sin(4t) = 0 \quad \Longleftrightarrow \quad t = \frac{k\pi}{4}, k \in [\![0, 4]\!]$ Corresponding points M_k: (cylindrical coordinates) $\overrightarrow{OM_k} = 2\vec{e_r} + 4cos(k\pi)\vec{e_z} = 2\vec{e_r} + 4.(-1)^k\vec{e_z}$ (Cartesian coordinates) $\overrightarrow{OM_k} = 2cos(\frac{k\pi}{2})\vec{e_x} + 2sin(\frac{k\pi}{2})\vec{e_y} + 4cos(k\pi)\vec{e_z}$ $\Longrightarrow \begin{cases} \overrightarrow{OM_k} = 2.(-1)^p\vec{e_x} + 4\vec{e_z} \text{ if } k = 2p \\ \overrightarrow{OM_k} = 2.(-1)^p\vec{e_y} - 4\vec{e_z} \text{ if } k = 2p+1 \end{cases}$	0.5 (cylindrical) 0.5 (Cartesian)
5. It is impossible to find t such that $\begin{cases} x(t) = 2cos(2t) = \sqrt{2} \\ y(t) = 2sin(2t) = \sqrt{2} \\ z(t) = 4cos(4t) = 4 \end{cases}$ So the point of Cartesian coordinates $(\sqrt{2}; \sqrt{2}; 4)$ does not belong to the curve	0.25
6. Curve b) is the trajectory	0.25
7. Cartesian coordinates of $M(t = \frac{\pi}{4}) \ : \ (x = 0; y = 2; z = -4)$ spherical coordinates of $M(t = \frac{\pi}{4}) \ : \ (\rho = 2\sqrt{5}; \phi = \frac{\pi}{2} + Arctan(2); \theta = \frac{\pi}{2})$	0.5 0.5
8. scheme	0.5

Exercise 4 - Differential equations (6 points)

1. We look for a particular solution $z_p(t) = A.e^{i\omega t}$ z_p satisfies the equation iff $A = \frac{U_0}{1+iRC\omega}$ Hence $z_p(t) = \frac{U_0}{1+iRC\omega}.e^{i\omega t}$	1
2. Associated homogeneous equation: $RCz' + z = 0$ The general solution is $z_h(t) = K.e^{rt}$ with r solution of the characteristic equation $RCr + 1 = 0$ Hence $r = \frac{-1}{RC}$ and $z_h(t) = K.e^{-\frac{t}{RC}}$	1
3. $z = z_p + z_h = \frac{U_0}{1+iRC\omega}.e^{i\omega t} + K.e^{-\frac{t}{RC}}$	0.5
4. We look for a particular solution $z_p(t) = A.e^{i\omega t}$ z_p satisfies the equation iff $A = \frac{U_0}{1-LC\omega^2}$ (if $\omega \neq \frac{1}{\sqrt{LC}}$) Hence $z_p(t) = \frac{U_0}{1-LC\omega^2}.e^{i\omega t}$ (if $\omega \neq \frac{1}{\sqrt{LC}}$) If $\omega = \frac{1}{\sqrt{LC}}$, we look for a particular solution $z_p(t) = At.e^{i\omega t}$ In this case, z_p satisfies the equation iff $A = \frac{U_0}{2i\omega}$ and $z_p = \frac{U_0}{2i\omega}.t.e^{i\omega t}$	1 (case $\omega \neq \frac{1}{\sqrt{LC}}$) 1 (case $\omega = \frac{1}{\sqrt{LC}}$)
5. Associated homogeneous equation: $LCz'' + z = 0$ The characteristic equation is $LCr^2 + 1 = 0$ Hence $r = \frac{\pm i}{\sqrt{LC}}$ and $z_h(t) = A.e^{\frac{it}{\sqrt{LC}}} + Be^{\frac{-it}{\sqrt{LC}}}$	1
6. $\begin{cases} z = \frac{U_0}{1-LC\omega^2}.e^{i\omega t} + A.e^{\frac{it}{\sqrt{LC}}} + Be^{\frac{-it}{\sqrt{LC}}} & \text{if} \quad \omega \neq \frac{1}{\sqrt{LC}} \\ z = \frac{U_0}{2i\omega}.t.e^{i\omega t} + A.e^{\frac{it}{\sqrt{LC}}} + Be^{\frac{-it}{\sqrt{LC}}} & \text{if} \quad \omega = \frac{1}{\sqrt{LC}} \end{cases}$	0.5

Exercise 5 - Analysis of a Lissajous curve (4 points + 1 bonus point)

1. if $t \in \left[0; \frac{\pi}{2}\right]$, $x = sin(2t) \in [0; 1]$	0.5 (scheme with scale)
2. $t = 0$: the point is $(0; 0)$ $t = \frac{\pi}{2}$: the point is $(0; -1)$	0.5
3. $\vec{T}(t) = x'(t)\vec{e_x} + y'(t)\vec{e_y} = 2cos(2t)\vec{e_x} + 3cos(3t)\vec{e_y}$ $\vec{T}$ is horizontal when $cos(3t) = 0$, *i.e.* $t = \frac{\pi}{6}$ or $t = \frac{\pi}{2}$ At that point $\vec{T} = 2cos(\frac{\pi}{3})\vec{e_x} = \vec{e_x}$ $\vec{T}$ is vertical when $cos(2t) = 0$, *i.e.* $t = \frac{\pi}{4}$ At that point, $\vec{T} = 3cos(\frac{3\pi}{4})\vec{e_y} = \frac{-3\sqrt{2}}{2}\vec{e_y}$	1 ($\vec{T}$) 2 (scheme completed with vectors of correct orientation)
4. $x(-t) = -x(t)$ and $y(-t) = -y(t) \Longrightarrow$ The portion of curve obtained for $t \in [-\pi; 0]$ is the symmetrical with respect to the x-axis of the portion of the curve obtained for $t \in [0; \pi]$. Moreover, for $t \in [0; \pi]$, $x(\pi - t) = -x(t)$ and $y(\pi - t) = y(t) \Longrightarrow$ The portion of curve obtained for $t \in \left[\frac{\pi}{2}; \pi\right]$ is the symmetrical with respect to the y-axis of the portion of the curve obtained for $t \in \left[0; \frac{\pi}{2}\right]$. 	Bonus +1