2$^{\text{nd}}$ MNTES exam – Semester 1
January 19$^{\text{th}}$, 2024. Duration: 2 h

No document allowed. No mobile phone, no electronic devices (like apple watch). No calculators allowed. The proposed grading scale is only indicative.
It is also reminded that the general clarity and cleanness of your paper may also be taken into account.

Exercise 1: Differential form (~ 2 points)

We consider the differential form ω, defined on $\mathbb{R}^3$ and expressed as:

$$\omega = yz^2dx + (xz^2 + z)dy + (2xyz + 2z + y)dz$$

1. Show that ω is closed.
2. Find the expression of $f(x, y, z)$ such that $\omega = df$, and $f(0, 0, 0) = 1$.

Exercise 2: Parametrization (~ 2 points)

We consider the ellipse shown in figure 1.

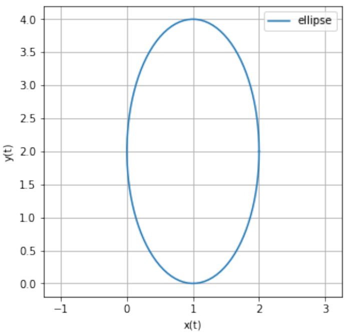

Figure 1: ellipse.

Propose a parametric equation of the ellipse using Cartesian coordinates.

Exercise 3: Kinematics (~ 6 points)

We consider a point object moving along the curve of following parametric equation (using cylindrical coordinates):

$$\begin{cases} r = 2 \\ z = 4\cos(2\theta) \\ \theta \in [0; 2\pi] \end{cases}$$

The angular velocity $\dot{\theta}$ constant equal to $\dot{\theta} = \frac{d\theta}{dt} = 2$. We will consider that $\theta = 0$ for $t = 0$.

1. Give a parametric description of the curve using cylindrical coordinates (that is to say express $r(t), \theta(t), z(t)$ and give the range for t).
2. Express the position vector $\overrightarrow{OM}$ in the local cylindrical frame, then in the cartesian frame.
3. Express the velocity $\vec{v}$ in the local cylindrical frame, and then in the Cartesian frame. Compute its norm.
4. Find the cylindrical coordinates of the points (on the trajectory) for which $\vec{v}$ is horizontal. What are the Cartesian coordinates of these points?

5. Does the point with Cartesian coordinates $(\sqrt{2}, \sqrt{2}, 4)$ belong to the curve?

6. Among all graphs in figure 2, find the one corresponding to the curve.

7. At time $t = \frac{\pi}{4}$, the object is at point M. Give the Cartesian and the spherical coordinates of this point. Represent on a scheme the local spherical frame at this point.

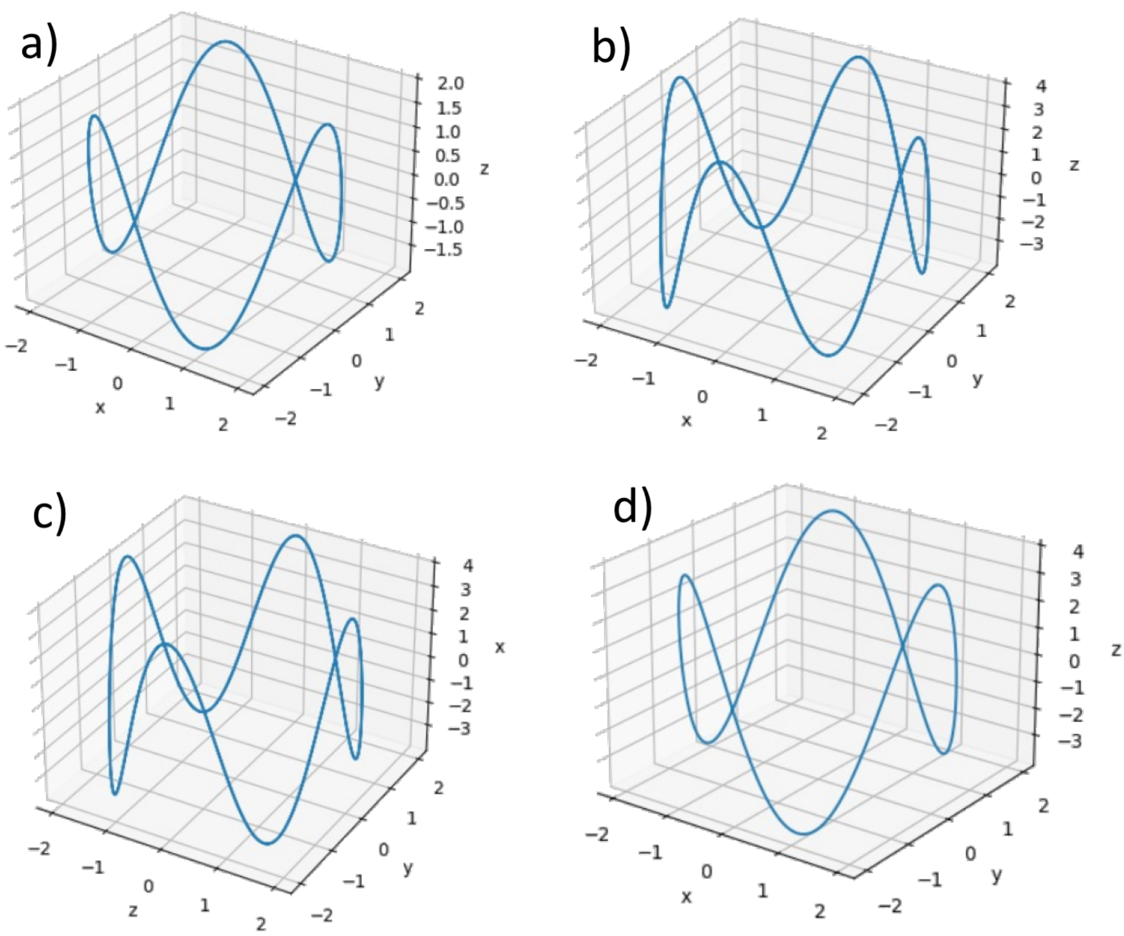

Figure 2: possible trajectories of point M.

Exercise 4: Differential equations ($\sim$ 6 points)
In this exercise, we study 2 different types of low-pass electronic filters.

RC filter
A RC filter is built as in figure 3, with R and C the resistance and the capacitance, respectively (positive constants).

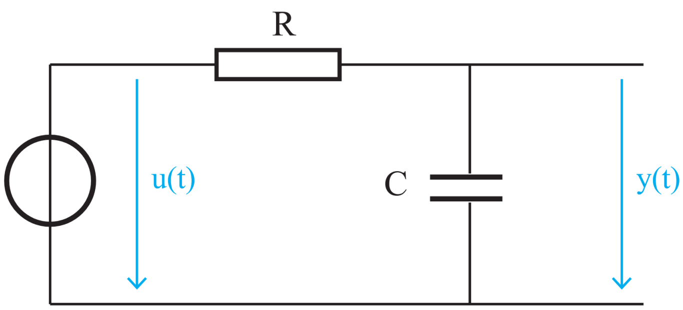

Figure 3: RC filter.

$u(t)$ is the input voltage and the filtered signal is $y(t)$. When $u(t)$ is a sinusoidal signal of angular frequency ω (positive constant), it is convenient to find the complex solutions $z(t)$ of the differential equation:

$$RCz' + z = U_0.e^{i\omega t} \quad \text{where} \quad i^2 = -1 \quad \text{and} \quad U_0 \in \mathbb{R}^{*+}$$

1. Find a particular solution $z_p(t)$.
2. Give the expression of the general solution $z_h(t)$ of the associated homogeneous equation.
3. Deduce the expression of the general solution $z(t)$ of the equation.

LC filter
The resistor is now replaced by an ideal coil of inductance L (positive constant). The new differential equation is:

$$LCz'' + z = U_0.e^{i\omega t}$$

4. Find a particular solution $z_p(t)$.
5. Give the expression of the general solution $z_h(t)$ of the associated homogeneous equation.
6. Deduce the expression of the general solution $z(t)$ of the equation.

Exercise 5: Analysis of a Lissajous curve (~ 4 points)

A parametric curve is defined in the frame $(O, \vec{e_x}, \vec{e_y})$ using the following parametric equation:

$$\begin{cases} x(t) = \sin(2t) \\ y(t) = \sin(3t) \\ t \in [-\pi; \pi] \end{cases}$$

Figure 4 shows the curve obtained when $t \in \left[0; \frac{\pi}{2}\right]$.

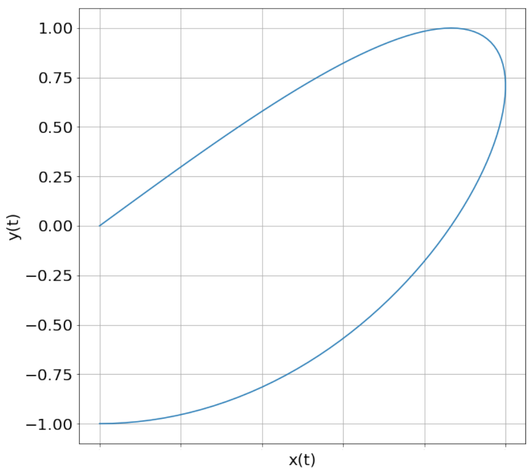

Figure 4: Lissajous curve.

1. From the minimum and maximum values of $x(t)$ on $[0; \pi/2]$, complete the scale on the x-axis.
2. Identify on the curve the points corresponding to $t = 0$ and $t = \frac{\pi}{2}$.
3. Give the expression of a tangent vector $\vec{T}(t)$ to this curve, for any value of $t \in [0, \pi/2]$. For which value(s) of $t \in [0, \pi/2]$ is this vector horizontal? Vertical? Place the corresponding points on the figure with their tangent vectors.
4. (Bonus) Considering the periodicity of the sine functions, draw schematically the curve on $[-\pi, \pi]$.