# Correction of the st MNTES exam – Semester 1
# November 17th, 2023

## Exercise 1: Complex numbers (8 points + 1 point bonus)

### Part A

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|---|---|
| 1. $z = \frac{2(-1+i)}{2-2i\sqrt{3}} = \frac{-1-\sqrt{3}}{4} + i\frac{1-\sqrt{3}}{4}$ | 1 |
| 2. $z = \frac{2(-1+i)}{2(1-i\sqrt{3})} = \frac{\sqrt{2}e^{i\frac{3\pi}{4}}}{2e^{-i\frac{\pi}{3}}} = \frac{\sqrt{2}}{2}e^{i\frac{13\pi}{12}} = \frac{\sqrt{2}}{2}e^{-i\frac{11\pi}{12}}$ | 2 |
| 3. $\tan(\frac{-11\pi}{12}) = \tan(\frac{\pi}{12}) = \frac{1-\sqrt{3}}{-1-\sqrt{3}} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$<br>$\tan\left(5\frac{\pi}{12}\right) = \frac{\sin\left(5\frac{\pi}{12}\right)}{\cos\left(5\frac{\pi}{12}\right)} = \frac{\cos\left(\frac{\pi}{12}\right)}{\sin\left(\frac{\pi}{12}\right)} = \frac{-1-\sqrt{3}}{1-\sqrt{3}} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3}$ | 1 for $\tan(\frac{\pi}{12})$ (with explanation) Bonus: 1 point for $\tan(\frac{5\pi}{12})$ |

### Part B

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| 1. With $z = \lvert z\rvert e^{i\theta}$, the equation becomes $\lvert z\rvert^5 e^{5i\theta} = 2$<br>Hence $\lvert z\rvert^5 = 2$ and $e^{5i\theta} = e^{i2k\pi}$ with $k \in \mathbb{Z}$<br>We obtain $z = 2^{\frac{1}{5}}e^{i\frac{2k\pi}{5}}$ with $k \in [\![0,4]\!]$ | 0.5 (modulus)<br>1 (argument) |
| 2. $\Delta = 16(-1+i)$<br>System of equations to find $\delta = a+ib$<br>Let's find $a, b$ so that $\delta = a+ib$ and $\delta^2 = (a+ib)^2 = -1+i$.<br>$\begin{cases} a^2-b^2=-1 \\ 2ab=1 \\ a^2+b^2=\sqrt{2} \end{cases} \iff \begin{cases} 2a^2 = \sqrt{2}-1 \\ 2b^2 = \sqrt{2}+1 \end{cases} \implies a = \frac{\sqrt{\sqrt{2}-1}}{\sqrt{2}}$ and $b = \frac{\sqrt{\sqrt{2}+1}}{\sqrt{2}}$ so<br>$\delta = 4\frac{\sqrt{\sqrt{2}-1}}{\sqrt{2}} + 4i\frac{\sqrt{\sqrt{2}+1}}{\sqrt{2}}$<br>Other method: $16(-1+i) = 4^2\sqrt{2}e^{i\frac{3\pi}{4}} = (4 \times 2^{\frac{1}{4}}e^{i\frac{3\pi}{8}})^2$<br>Finally, $z = -\frac{i}{2} \pm \left(\frac{\sqrt{\sqrt{2}-1}}{2\sqrt{2}} + i\frac{\sqrt{\sqrt{2}+1}}{2\sqrt{2}}\right)$ | 0.5 ($\Delta$)<br>1.5 ($\delta$)<br>0.5 (solutions) |

**Exercise 2 - Vectors** (14 points)

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| 1. $\overrightarrow{A_1A_2} = (a, 0, 0)$<br>$\overrightarrow{A_1A_3} = (0, a, 0)$<br>$\overrightarrow{A_1A_5} = (0, 0, a)$ | 3*0.5 |
| 2. $\left[\overrightarrow{A_1A_2}, \overrightarrow{A_1A_3}, \overrightarrow{A_1A_5}\right] = a^3$. It is the volume of the cube | 1 + 0.5 |
| 3a. $C(\frac{a}{2}; \frac{a}{2}; \frac{a}{2})$ | 0.5 |
| 3b. $\vec{q} = (0; -a^2/2; a^2/2)$ | 1 |
| 3c. $\vec{q}$ points upwards (use right-hand rule or using sign of the third component from previous question) | 0.5 (direction) + 0.5 (rule) |
| 3d. $\vec{q} \cdot \overrightarrow{A_2A_4} = -\frac{a^3}{2}$<br>$\Vert\vec{q}\Vert = \frac{a^2}{\sqrt{2}}$ and $\Vert\overrightarrow{A_2A_4}\Vert = a$<br>and $\cos(\theta) = \frac{\vec{q} \cdot \overrightarrow{A_2A_4}}{\Vert\vec{q}\Vert \Vert\overrightarrow{A_2A_4}\Vert} = -\frac{\sqrt{2}}{2}$<br>so we deduce $\theta = \frac{3\pi}{4}$ | 1 (use of dot product or cross product)<br>1.5 (calculation) |
| 3e. Let $\vec{u}$ be the projection of $\vec{q}$ onto $\overrightarrow{A_2A_4}$<br>$\vec{u} = \left(\frac{\vec{q} \cdot \overrightarrow{A_2A_4}}{\Vert\overrightarrow{A_2A_4}\Vert}\right) \frac{\overrightarrow{A_2A_4}}{\Vert\overrightarrow{A_2A_4}\Vert} = (\Vert\vec{q}\Vert \cos(\theta))\,(0; 1; 0) = (0; -\frac{a^2}{2}; 0)$ | 1 + 0.5 |
| 4. $\overrightarrow{A_1G_1} = \frac{1}{8} \sum \overrightarrow{A_1A_i}$<br>With the components of the vectors, we obtain $G_1 = (\frac{a}{2}; \frac{a}{2}; \frac{a}{2})$<br>$G_1 = C$ (obvious as centroid = center) | 0.5<br>0.5<br>0.5 |
| 5a. $\overrightarrow{A_1G_2} = \frac{1}{6}(\overrightarrow{A_1A_1} + \overrightarrow{A_1A_2} ... + \overrightarrow{A_1A_5} + \overrightarrow{A_1A_8})$<br>With the components, we obtain $G_2 = (\frac{a}{2}; \frac{a}{2}; \frac{a}{3})$ | 0.5<br>0.5 |
| 5b. $G_1G_2 = \Vert\overrightarrow{G_1G_2}\Vert = \frac{a}{6}$ | 0.5 |
| 6. We can take $\vec{n_1} = \overrightarrow{A_1A_2} \times \overrightarrow{A_1G_1}$<br>Hence $\vec{n_1} = \vec{q}(0; -\frac{a^2}{2}; \frac{a^2}{2})$<br>other valid option: find $\vec{n} = (n_1, n_2, n_3)$ so that $\vec{n} \cdot \overrightarrow{A_1A_2} = 0$ and $\vec{n} \cdot \overrightarrow{A_1G1} = 0$<br>which leads to $\begin{cases} n_1 = 0 \\ n_2 + n_3 = 0 \end{cases}$ so $\vec{n} = (0, \lambda, -\lambda)$, $\lambda \in \mathbb{R}$ is a normal vector to the plane | 1<br>0.5 |

**Exercise 3 - Differential calculus (10 points + 3 points bonus)**

**Part A**

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| 1. $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = y.x^{y-1}dx + Ln(x).e^{yLnx}dy$<br>$d_{(x_0,y_0)}f = \frac{\partial f}{\partial x}(x_0,y_0)dx + \frac{\partial f}{\partial y}(x_0,y_0)dy = y_0.x_0^{y_0-1}dx + \ln(x_0).e^{y_0 \ln x_0}dy$<br>With $x_0 = 1$ and $y_0 = 2$, we get $df_{x_0;y_0} = 2dx$ | 1<br>0.5 |
| 2. Small variations: $f(x_0+dx; y_0+dy) \approx f(x_0;y_0) + df_{x_0;y_0}(\delta x;\delta y) \approx x_0^{y_0} + y_0.x_0^{y_0-1}\delta x + Ln(x_0).e^{y_0 Lnx_0}\delta y$ | 0.5 |
| 3. We take $x_0 = 1$; $y_0 = 2$; $\delta x = 0.04$ and $\delta y = 0.02$<br>and we obtain $1.04^{2.02} \approx 1.08$ | 0.5 |

**Part B**

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|---|---|
| 1. Scheme f a parallelogram with the points and the angle labelled | 0.5 |
| 2. $S(a,c,\theta) = ac\sin\theta$ | 0.5 |
| 3. $dS = \frac{\partial S}{\partial a}da + \frac{\partial S}{\partial c}dc + \frac{\partial S}{\partial \theta}d\theta = c\sin\theta da + a\sin\theta dc + ac\cos\theta d\theta$ | 1 |
| 4. $\delta S \approx c\sin\theta\delta a + a\sin\theta\delta c$ | 0.5 |
| 5. $V = S.\ell = ac\ell\sin\theta$<br>$m = \rho V = ac\ell\rho_0\sin\theta(1+e^{-\frac{\ell}{\ell_0}})$ | 0.5<br>0.5 |
| 6. Assuming $0 < \theta < \frac{\pi}{2}$ then<br>$\Delta S = c\sin\theta\Delta a + a\sin\theta\Delta c + ac\cos\theta\Delta\theta$<br>$dV = Sd\ell + \ell dS = \ell c\sin\theta da + \ell a\sin\theta dc + \ell ac\cos\theta d\theta + ac\sin\theta d\ell$<br>so $\Delta V = \ell c\sin\theta\Delta a + \ell a\sin\theta\Delta c + \ell ac\cos\theta\Delta\theta + ac\sin\theta\Delta\ell$<br>$dm = \rho(\ell)dV + Vd\rho(\ell) = c\ell\rho_0\sin\theta(1+e^{-\frac{\ell}{\ell_0}})da + a\ell\rho_0\sin\theta(1+e^{-\frac{\ell}{\ell_0}})dc + ac\rho_0\sin\theta(1+\frac{\ell_0-\ell}{\ell_0}e^{-\frac{\ell}{\ell_0}})d\ell + ac\ell\rho_0\cos\theta(1+e^{-\frac{\ell}{\ell_0}})d\theta$<br>so $\Delta m = c\ell\rho_0\sin\theta(1+e^{-\frac{\ell}{\ell_0}})\Delta a + a\ell\rho_0\sin\theta(1+e^{-\frac{\ell}{\ell_0}})\Delta c + ac\rho_0\sin\theta(1+\frac{\vert\ell_0-\ell\vert}{\ell_0}e^{-\frac{\ell}{\ell_0}})\Delta\ell + ac\ell\rho_0\cos\theta(1+e^{-\frac{\ell}{\ell_0}})\Delta\theta$ | 0.5 ($\Delta S$)<br>1 ($\Delta V$)<br>1 ($\Delta m$) |
| 7. $S = (43.3 \pm 7.0)\ cm^2$<br>$V = (13.0 \pm 2.4)\ cm^3$<br>$m = (13.6 \pm 3.0)\ g$ | 0.5 (S) + bonus 1 ($\Delta S$)<br>0.5 (V) + bonus 1 ($\Delta S$)<br>0.5 (m) + bonus 1 ($\Delta S$) |