1^{st} MNTES exam – Semester 1
November 17^{th}, 2023. Duration: 1.5 h

No document allowed. No mobile phone, no electronic devices (like apple watch). Calculators in exam mode allowed. The proposed grading scale is only indicative.
It is also reminded that the general clarity and cleanness of your paper may also be taken into account.

Exercise 1: Complex numbers (5 points)

Both parts are independent from each other.

Part A

We consider the complex number $z = \frac{2(-1+i)}{(\sqrt{3}-i)^2}$

1. Express z in the Cartesian form.
2. Give the exponential form of z.
3. Deduce the value of $tan(\frac{5\pi}{12})$.

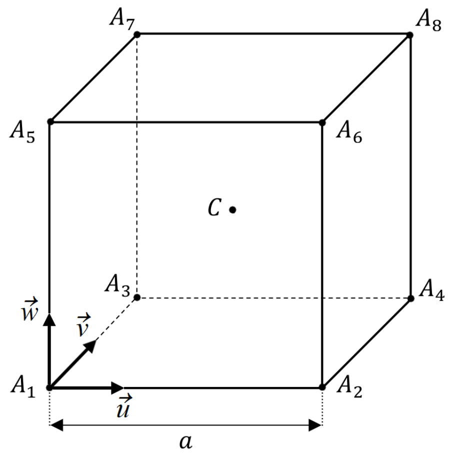
Part B

Solve in $\mathbb{C}$ the following equations:

1. $z^5 = 2$
2. $4z^2 + 4iz - i = 0$

Exercise 2 - Vectors (9 points)

Let be a cube with side $a > 0$ and vertices A_i for $i \in [1, 8]$. Note C the center of the cube.
The unit vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ are introduced. respectively of the same direction as $\overrightarrow{A_1A_2}$, $\overrightarrow{A_1A_3}$ and $\overrightarrow{A_1A_5}$.
Throughout the exercise, we will work in the direct orthonormal reference frame $(A_1, \vec{u}, \vec{v}, \vec{w})$ and express all answers in terms of a.



1. Determine the components of the vectors $\overrightarrow{A_1A_2}$, $\overrightarrow{A_1A_3}$ and $\overrightarrow{A_1A_5}$.
2. Calculate the triple scalar product $\left[\overrightarrow{A_1A_2}, \overrightarrow{A_1A_3}, \overrightarrow{A_1A_5}\right]$. What does this quantity correspond to, for the cube?
3. (a) Give the coordinates of point C.
   (b) Calculate the vector product: $\vec{q} = \overrightarrow{A_1A_2} \times \overrightarrow{A_1C}$.
   (c) Does the vector $\vec{q}$ point downwards or upwards with respect to the plane (P) defined by the points A_1, A_2 and A_3? Justify.

(d) What is the value of the non-oriented angle between $\vec{q}$ and $\overrightarrow{A_2A_4}$?

(e) Give the components of the projection of $\vec{q}$ in the direction of $\overrightarrow{A_2A_4}$.

4. By calculation, give the coordinates of the centroid G_1 of the points A_i for $i \in [1, 8]$. Is what you have found surprising? Why or why not?

5. In this question, we assume that the weights of points A_6 and A_7 are zero.

   (a) Calculate the coordinates of the new centroid G_2.

   (b) Deduce the distance between G_1 and G_2.

6. Consider the plane (P_1) defined by the points (A_1, A_2, G_1). Determine a normal vector $\vec{n}_1$ to (P_1).

Exercise 3 - Differential calculus (6 points)

Both parts are independent from each other.

Part A

We consider the function $f(x, y) = x^y = e^{y\ln(x)}$, where x is a positive real and y a real.

1. Compute the value of its differential df at the point $x_0 = 1$ and $y_0 = 2$.

2. Small variations: give the literal expression of δf as a function of x_0, y_0, δx and δy.

3. Deduce the value of $1.04^{2.02}$ to the nearest hundredth.

Part B

Consider a metal plate cut into the shape of a parallelogram $OABC$ with sides $OA = a$, $OC = c$ and angle $\theta = (\vec{OA}, \vec{OC})$.

1. Make a scheme.

2. Determine the function $S(a, c, \theta)$ used to calculate the area of the parallelogram.

3. Express the differential of S.

4. compute the variation δS of the function S when a is changed by δa, c by δc and θ remains constant.

5. The plate has a thickness ℓ and a mass density $\rho(\ell) = \rho_0(1 + e^{-\ell/\ell_0})$, with ρ_0 and ℓ_0 positive constants. Express its volume V and mass m.

6. a, c, ℓ, θ are measured with the respective uncertainties $\Delta a, \Delta c, \Delta\ell, \Delta\theta$. Express the absolute uncertainties ΔS, ΔV and Δm.

7. Numerical application: We measure: $a = (5.0 \pm 0.5)\ cm$, $c = (10 \pm 0.5)\ cm$, $\ell = (3.0 \pm 0.1)\ mm$ and $\theta = (60 \pm 1)°$. We take $\rho_0 = 1\ g.cm^{-3}$ and $\ell_0 = 1\ mm$, known without uncertainty. Give the values of S, V and m with their uncertainties, and write the results in format "(... ±...) *unit*".