1st MONTy exam – Semester 1
November 22th, 2024. Duration: 1.5 h

No document allowed. No mobile phone, no electronic devices (like apple watch). No calculator. The proposed grading scale is only indicative.
It is also reminded that the general clarity and cleanness of your paper may also be taken into account.

Exercise 1 - Quadratic equations ($\approx$ 3 points)

Solve in $\mathbb{C}$ the following equations (we expect details of the steps, not just the results):

1. $z^2 = 1 + i$
2. $z^2 + \sqrt{3}z - i = 0$

Exercise 2 - Powers of a complex number ($\approx$ 2 points)

Consider the complex number $\omega = \dfrac{-1+i}{\sqrt{2}}$.

1. Write ω in the trigonometric form.
2. Sketch the position in $\mathbb{C}$ of the complex numbers ω^n for $n = 1, 2, \ldots, 8$.
3. What is ω^{2024} worth?
4. Consider the complex number $z = \frac{\omega}{\sqrt{2}}$. On the previous sketch, place the complex numbers z^2, z^4, z^8, and z^{-2}.

Exercise 3 - Vectors ($\approx$ 6 points)

Consider the space $\mathbb{R}^3$ with the orthonormal frame $(O, \vec{e}_x, \vec{e}_y, \vec{e}_z)$, and consider the four points

$$A = (1,1,0) \quad \text{and} \quad B = (0,2,0) \quad \text{and} \quad C = (0,1,2) \quad \text{and} \quad D = (2,0,2)\,.$$

1. Sketch the frame and and place the four points (approximately).
2. Give the components of a unit vector $\vec{u}$ such that $(\overrightarrow{AB}, \overrightarrow{AC}, \vec{u})$ forms a direct frame.
3. Compute the components of $\overrightarrow{AH}$, where H is the orthogonal projection of D onto the line (Δ) passing through A and of directing vector $\vec{u}$.
4. Is D on the same side of plane (ABC) as O? (Give the details of your method)
5. Express the "geometric" angle (between 0 and π) between the vectors $\vec{u}$ and $\overrightarrow{AD}$.
6. Can the point $M(2,0,0)$ be the center of mass of of (A, a) and (B, b) with $a, b \in \mathbb{Z}$? (justify your answer with a calculation).
7. Without any calculation and using the associativity of the centers of mass, indicate the position of the centroid G of the points A, B, C, D.

Exercise 4 - Differential calculus (≈ 4 points)

1. Express the differential of the function

$$f : \mathbb{R}^3 \to \mathbb{R}\,, \quad (x, y, z) \mapsto \cos(x + y^2)e^{-z}\,,$$

2. Is the following differential form, defined on $\mathbb{R}^2$, closed? If it is the case, express the functions f such that $\omega = df$.

$$\omega = e^x(y + 5)dx + (e^x + 3e^y)dy$$

Exercise 5 - Uncertainties and variations (≈ 5 points)

The capacitance of a cylindrical capacitor is expressed as: $C = \frac{2\pi\varepsilon h}{ln(\frac{R_2}{R_1})}$ where R_1 and $R_2 > R_1$ are respectively the inner and the outer diameters of the the dielectric material inside the capacitor, h its height, and ε is a positive constant (see figure 1).

1. We assume that R_1 and R_2 are very close to each other, with $R_2 = R_1 + e$. Show that the capacitance is approximately equal to $C = \frac{\varepsilon S}{e}$, S being the area of the inner cylinder forming the capacitor.
2. Using the expression $C = \frac{2\pi\varepsilon h}{ln(\frac{R_2}{R_1})}$, give an approximated expression of the variation δC of the capacitance if R_1, R_2, h are reduced by $\delta R_1, \delta R_2$ and δh, respectively.
3. Using again the expression $C = \frac{2\pi\varepsilon h}{ln(\frac{R_2}{R_1})}$, compute the uncertainty ΔC in the case $\Delta h = a$ and $\Delta R_1 = \Delta R_2 = \frac{a}{2}$.

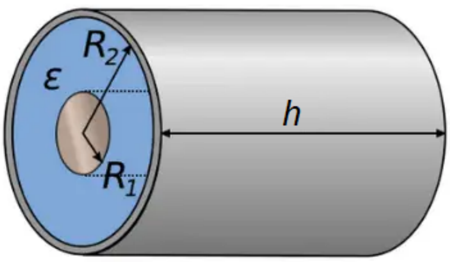

Figure 1: scheme of a cylindrical capacitor