INSTITUT NATIONAL DES SCIENCES APPLIQUÉES DE LYON

Département du Premier Cycle - SCAN - First

MTES test 2 - Duration 2 h

Warnings and advices
- **Documents, dictionaries, phones, and calculators are FORBIDDEN.**
- **The marking scheme is only given as an indication.**

Exercise 1 *(4 pts)*

Consider the surface $S = \{(x,y,z) \in \mathbb{R}^3 \,|\, x^2+y^2+z^2 = 4, x \geqslant 0, y \geqslant 0, z \leqslant 0\}$ with the natural orientation of the sphere.
Give a representation of this surface and compute the flux of the vector field

$$\overrightarrow{A}(x,y,z) = z\overrightarrow{e_y} + x\overrightarrow{e_z}$$

through the surface S.

Exercise 2 *(5 pts)*

We consider here three surfaces that make the boundary of a volume V.

- $D_1 = \{(x,y,z) \in \mathbb{R}^3 | x^2+y^2 \leqslant 4 \quad \text{and} \quad z = 0\}$
- $D_2 = \{(x,y,z) \in \mathbb{R}^3 | x^2+y^2 \leqslant 4 \quad \text{and} \quad z = 1\}$
- $C = \{(x,y,z) \in \mathbb{R}^3 | x^2+y^2 = 4 \quad \text{and} \quad 0 \leqslant z \leqslant 1\}$

The three surfaces have the natural orientation given by V.
We consider on $\mathbb{R}^3$ the vector field

$$\overrightarrow{A}(x,y,z) = 2x\overrightarrow{e_x} + (1-z)\overrightarrow{e_z}$$

1. Represent the volume V specifying the surfaces D_1, D_2 and C.
2. Compute the flux of $\overrightarrow{A}$ through D_1.
3. Compute the flux of $\overrightarrow{A}$ through D_2.
4. We consider the scalar field Φ on $\mathbb{R}^3$ given in cartesian coordinates by $\Phi(x,y,z) = x^2+y^2$.
   ? (a) Compute the gradient of Φ in the cartesian frame and in cartesian coordinates.
   ? (b) ~~Compute the gradient of Φ in the cylindrical frame and in cylindrical coordinates.~~
   ? (c) **Deduce** from the previous questions the normal normalized vector field $\overrightarrow{n}$ on C that defines the orientation of C.
   (d) Compute the flux of $\overrightarrow{A}$ through C.

? 5. Compute the flux of $\overrightarrow{A}$ through the boundary of V.

EXERCISE 3 *(5 pts)*
We consider the vector field $\overrightarrow{F}$ defined on $\mathbb{R}^2$ by :

$$\overrightarrow{F}(x,y) = x\overrightarrow{e_x} - y\overrightarrow{e_y}$$

1. Compute the field lines of $\overrightarrow{F}$. Draw some of them.
2. Show without finding a potential that $\overrightarrow{F}$ is derived from a potential V.
3. Find the potential V such that $V(0,0) = 0$.
4. Draw on the same picture of question 1 the level curves $V^{[-1]}(\{k\})$ of V. No justification required **BUT just specify which ones are for $k > 0$, $k = 0$ and $k < 0$.**
5. (a) Show that the points $A(0,0)$ and $B(1,1)$ are on the same level curves of V.
   (b) Compute the circulation of $\overrightarrow{F}$ from A to B along the parabola of equation $y = x^2$.

EXERCISE 4 *(6 pts)*
We consider the three vector fields defined on $\mathbb{R}^3 \setminus Oz$ given in **cylindrical coordinates** and in the local **cylindrical frame** :

- $\overrightarrow{A} = \dfrac{2\cos(\theta)}{r^3}\overrightarrow{e_r}$
- $\overrightarrow{B} = \dfrac{\sin(\theta)}{r^3}\overrightarrow{e_\theta}$
- $\overrightarrow{S} = \overrightarrow{A} + \overrightarrow{B} + \overrightarrow{\nabla}(\Psi)$ where Ψ is a C^2 scalar field on $\mathbb{R}^3$

We consider the points : $E(1,0,0), F(2,0,0), G(0,2,0)$ and $H(0,1,0)$ and the curve Γ oriented in the direction $EHGF$ made of :
- The arc of circle centered at 0 of radius 1 from E to H.
- The segment line $[HG]$.
- The arc of circle centered at 0 of radius 2 from G to F.
- The segment line $[FE]$.

1. Draw a picture of Γ.
2. (a) On which parts of Γ the infinitesimal circulation of $\overrightarrow{A}$ is 0? Justify your answers with as few computations as possible.
   (b) On which parts of Γ the infinitesimal circulation of $\overrightarrow{B}$ is 0? Justify your answers with as few computations as possible.
3. Compute the circulation of $\overrightarrow{A}$ along Γ. From this result, can we deduce if $\overrightarrow{A}$ is derived or not from a potential?
4. Compute the circulation of $\overrightarrow{B}$ along Γ. From this result, can we deduce if $\overrightarrow{B}$ is derived or not from a potential?
5. Compute the circulation of $\overrightarrow{S}$ along Γ. From this result, can we deduce if $\overrightarrow{S}$ is derived or not from a potential?
