INSTITUT NATIONAL DES SCIENCES APPLIQUÉES DE LYON
PÔLE DE MATHÉMATIQUES

INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

Département du Premier Cycle - SCAN1 - **2017-2018**

MTES Final Test

Warnings and advices

- **Documents, dictionaries, phones, and calculators are FORBIDDEN.**
- **The marking scheme is only given as an indication.**

Exercise 1 *(2.5 pts)*

In $\mathbb{R}^3$, with the standard frame, we consider the vector field $\overrightarrow{F} = yz\overrightarrow{e_x} + xz\overrightarrow{e_y} + xy\overrightarrow{e_z}$.

1. Prove that $\overrightarrow{F}$ is derived from a potential.
2. Find the expression of the potential $P(x,y,z)$ of $\overrightarrow{F}$ such that $P(0,0,0) = 0$.
3. Determine the equation of the level sets of the potential $P(x,y,z)$.
4. In the plane Oxy, we consider the square $ABCD$ with $A(-1,-1,0)$, $B(-1,1,0)$, $C(1,1,0)$ et $D(1,-1,0)$. The orientation of the boundary of the square is $ABCD$. Compute the flux of $\overrightarrow{F}$ through the square $ABCD$.

Exercise 2 *(6 pts)*

In $\mathbb{R}^2$, we consider the vector field $\overrightarrow{F} = \overrightarrow{e_x} + \sin(x)\overrightarrow{e_y}$.

1. On a field map, represent the vector field at the following 8 points :

$$(0,0), (0,\frac{\pi}{2}), (0,\pi), (0,\frac{3\pi}{2}), (\frac{\pi}{2},0), (\pi,0), (\frac{3\pi}{2},0), (\pi,\pi)$$

2. (a) Compute the circulation of $\overrightarrow{F}$ along the segment line $[M_1M_2]$ with $M_1(0,0)$ and $M_2(\frac{\pi}{2},1)$.

   (b) Let $M_3(\frac{\pi}{2},0)$. Compute the circulations of $\overrightarrow{F}$ along the segment lines $[M_1M_3]$ and $[M_3M_2]$. What can you conclude ?

3. (a) Determine the field lines.

   (b) Draw the field line that contains the origin.

   (c) Check, by computation, that the points $M_1(0,0)$ and $M_2(\frac{\pi}{2},1)$ are on the same field line that we will denote Γ.

   (d) Compute the circulation from M_1 to M_2 of $\overrightarrow{F}$ along Γ.

Exercise 3 *(3 pts)*

Compute the outgoing flux of the vector field $\overrightarrow{F} = \overrightarrow{e_z}$ through the surface

$$S = \{(x,y,z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = R^2 \text{ and } z \geqslant 0\}$$

Exercise 4 (8.5 *pts*)

We consider a Piece of Cone (PC) which is a surface defined in cylindrical coordinates by the equations

$$H - z = \frac{rH}{R} \text{ and } z \in [0, \frac{H}{2}]$$

where H and R are strictly positive constants.
we consider the vector field $\overrightarrow{F_1}$ defined on $\mathbb{R}^3$ **minus the** O_z **axis** by

$$\overrightarrow{F_1} = \overrightarrow{e_r} + \cos(\theta)\overrightarrow{e_\theta} + \overrightarrow{e_z}$$

in cylindrical coordinates and in the cylindrical local frame.

1. Represent the surface (PC) and indicate on the scheme the quantities H and R.
2. (a) Give the expression of the gradient in cylindrical coordinates.(No proof required!).
   (b) Deduce that $\frac{H}{R}\overrightarrow{e_r} + e_z$ is a normal to (PC).
   (c) Show that if we parametrize (PC) with θ and z then an infinitesimal element of surface is $dS = \frac{R(H-z)}{H}\sqrt{1 + \frac{R^2}{H^2}}\, d\theta\, dz$.
   (d) Compute the outgoing flux of $\overrightarrow{F_1}$ through (PC).
3. Consider the parametrized curve Γ :

$$\begin{cases} r(t) &= \frac{tR}{2} \\ \theta(t) &= 2\pi t \\ z(t) &= H\frac{2-t}{2} \end{cases} \qquad t \in [1, 2]$$

$$\frac{\frac{tR}{2} H}{R}$$

$$H - \frac{tRH}{2R} = H$$

   (a) Justify that this curve is indeed on PC.
   (b) Sketch this curve on your representation of (PC) at question 1.
   (c) Show that $\overrightarrow{F_1}$ is not derived from a potential.
   (d) Compute the circulation of $\overrightarrow{F_1}$ on Γ with the natural orientation given by its parametrization.
4. (a) Determine a function $\alpha(\theta)$ (that only depends on θ) such that the vector field

$$\overrightarrow{F_2} = \alpha(\theta)\overrightarrow{e_r} + \cos(\theta)\overrightarrow{e_\theta} + \overrightarrow{e_z}$$

   is derived from a potential φ **on** $\mathbb{R}^3$ **minus the** O_z **axis** that you will determine.
   (b) Deduce the circulation of $\overrightarrow{F_2}$ on Γ.