INSTITUT NATIONAL DES SCIENCES APPLIQUÉES DE LYON

Département du Premier Cycle - SCAN - First

Answers IE 1- 26/04/18 MTES

Exercise 1

1. (a) See at the end.

   (b) – $\mathcal{C}_1 : (x-1)^2 + (y-1)^2 = 1$
   – $\mathcal{C}_2 : x^2 + (y-1)^2 = 2$

2. (a) See at the end.

   (b) Yes and yes.

   (c) From the drawing we see that $x \in [0,1]$ and $y \in [1-\sqrt{2}, 1]$.
   Now for each x.
   - The maximal value of y is when it is on $\mathcal{C}_1$. Which means $(y-1)^2 = 1-(x-1)^2$. Since $y \leqslant 1$ then $y-1 \leqslant 0$ so $y-1 = -\sqrt{1-(x-1)^2}$ which means the maximal value for y is $1-\sqrt{1-(x-1)^2}$.
   - The minimal value of y is when it is on $\mathcal{C}_2$. Which means $(y-1)^2 = 2-x^2$. Since $y \leqslant 1$ then $y-1 \leqslant 0$ so $y-1 = -\sqrt{1-x^2}$ which means the minimal value for y is $1-\sqrt{2-x^2}$.

   Thus $D = \{(x,y) | 0 \leqslant x \leqslant 1, 1-\sqrt{2-x^2} \leqslant y \leqslant 1-\sqrt{1-(x-1)^2}\}$.

3. (~~a~~) 1 We use $\frac{x}{\sqrt{2}} = \cos(\theta)$. So $\int_0^1 \sqrt{2-x^2}\, dx = \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \sqrt{2}\sqrt{1-\cos^2(\theta)}(-\sqrt{2}\sin(\theta)\, d\theta) = 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^2(\theta)\, d\theta = 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1-\cos(2\theta)}{2}\, d\theta = [\theta - \frac{\sin(2\theta)}{2}]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{\pi}{4} + \frac{1}{2}$, since $\sin \geqslant 0$ on $[\frac{\pi}{4}, \frac{\pi}{2}]$.

   (b) *Area* $= \int_{x=0}^{1} \int_{y=1-\sqrt{2-x^2}}^{1-\sqrt{1-(x-1)^2}} dy\, dx = \int_{x=0}^{1} \sqrt{2-x^2} - \sqrt{1-(x-1)^2}\, dx = \frac{\pi}{4} + \frac{1}{2} - \frac{\pi}{4} = \frac{1}{2}$.

   (c) BONUS : Use $u = (x-1)$ so $\int_0^1 \sqrt{1-(x-1)^2}\, dx = \int_{-1}^{0} \sqrt{1-u^2}\, du$. We recognize the computation of the area of upper half disk of radius 1 centered at 0 for $x \in [-1, 0]$. One then gets a quarter of a disk, whose area is thus $\frac{\pi}{4}$.

Exercise 2

1. $M = \int_{r=0}^{R}\int_{\theta=0}^{2\pi} \sigma_0\theta(2\pi-\theta)r\ dr\ d\theta = \sigma_0(\int_{r=0}^{R} r\ dr)(\int_{\theta=0}^{2\pi}\theta(2\pi-\theta)\ d\theta) = \sigma_0\frac{R^2}{2}\frac{4\pi^3}{3} = \frac{2R^2\sigma_0\pi^3}{3}$

2. The disk is symmetric with respect to the Ox axis because this symmetry transforms the angle θ into $2\pi-\theta$ so σ becomes $\sigma_0(2\pi-\theta)(2\pi-(2\pi-\theta)) = \sigma_0(2\pi-\theta)\theta(2\pi-\theta) = \sigma$. So $G_y = 0$.

   And $MG_x = \int_{r=0}^{R}\int_{\theta=0}^{2\pi}\sigma_0\theta(2\pi-\theta)r^2\cos(\theta)\ dr\ d\theta = \sigma_0\frac{R^3}{3}(\int_{\theta=0}^{2\pi}\theta\cos(\theta)(2\pi-\theta)\ d\theta)$

   Using IPP, $\int_{\theta=0}^{2\pi}\theta\cos(\theta)\ d\theta = [\theta\sin(\theta)]_0^{2\pi} - \int_{\theta=0}^{2\pi}\sin(\theta)\ d\theta = 0$. And $\int_{\theta=0}^{2\pi}\theta^2\cos(\theta)\ d\theta = [\theta^2\sin(\theta)]_0^{2\pi} - 2\int_{\theta=0}^{2\pi}\theta\sin(\theta)\ d\theta = 0 - 2([-\theta\cos(\theta)]_0^{2\pi} + \int_{\theta=0}^{2\pi}\cos(\theta)\ d\theta = 4\pi$

   Thus $MG_x = \sigma_0\frac{-4\pi R^3}{3}$ and finally $G_x = \frac{-2}{\pi^2}R$.

Exercise 3

1.
2. (a)
   (b) We have $z = \ln(r)$ so $r = e^z$. And when $r = 1, z = 0$ and when $r = e$, $z = 1$.
   We have $r'(z) = e^z$ so $dS = \sqrt{1+e^{2z}}e^z\ dz\ d\theta$

   So $S = 2\pi\int_0^1\sqrt{1+e^{2z}}e^z\ dz$. We use the substitution $u = e^z$ and thus $S = 2\pi I$.

3. (a) $M = \int_0^{2\pi}\int_0^1\frac{1}{\sqrt{1+e^{2z}}}\sqrt{1+e^{2z}}e^z\ dz\ d\theta = 2\pi\int_0^1 e^z\ dz = 2\pi(e-1)$

   (b) We have a symmetry of revolution around Oz since σ depends only on z so $G_x = G_y = 0$.

   Now $MG_z = \int_0^{2\pi}\int_0^1\frac{1}{\sqrt{1+e^{2z}}}\sqrt{1+e^{2z}}ze^z\ dz\ d\theta = 2\pi\int_0^1 ze^z\ dz = 2\pi([ze^z]_0^1 - \int_0^1 e^z\ dz) = 2\pi$. Thus $G_z = \frac{1}{e-1}$.

4. $V = \int_0^{2\pi}\int_0^1\int_{r=0}^{e^z} r\ dr\ d\theta\ dz = 2\pi\int_0^1\frac{e^{2z}}{2}\ dz = \frac{e^2-1}{2}\pi$.

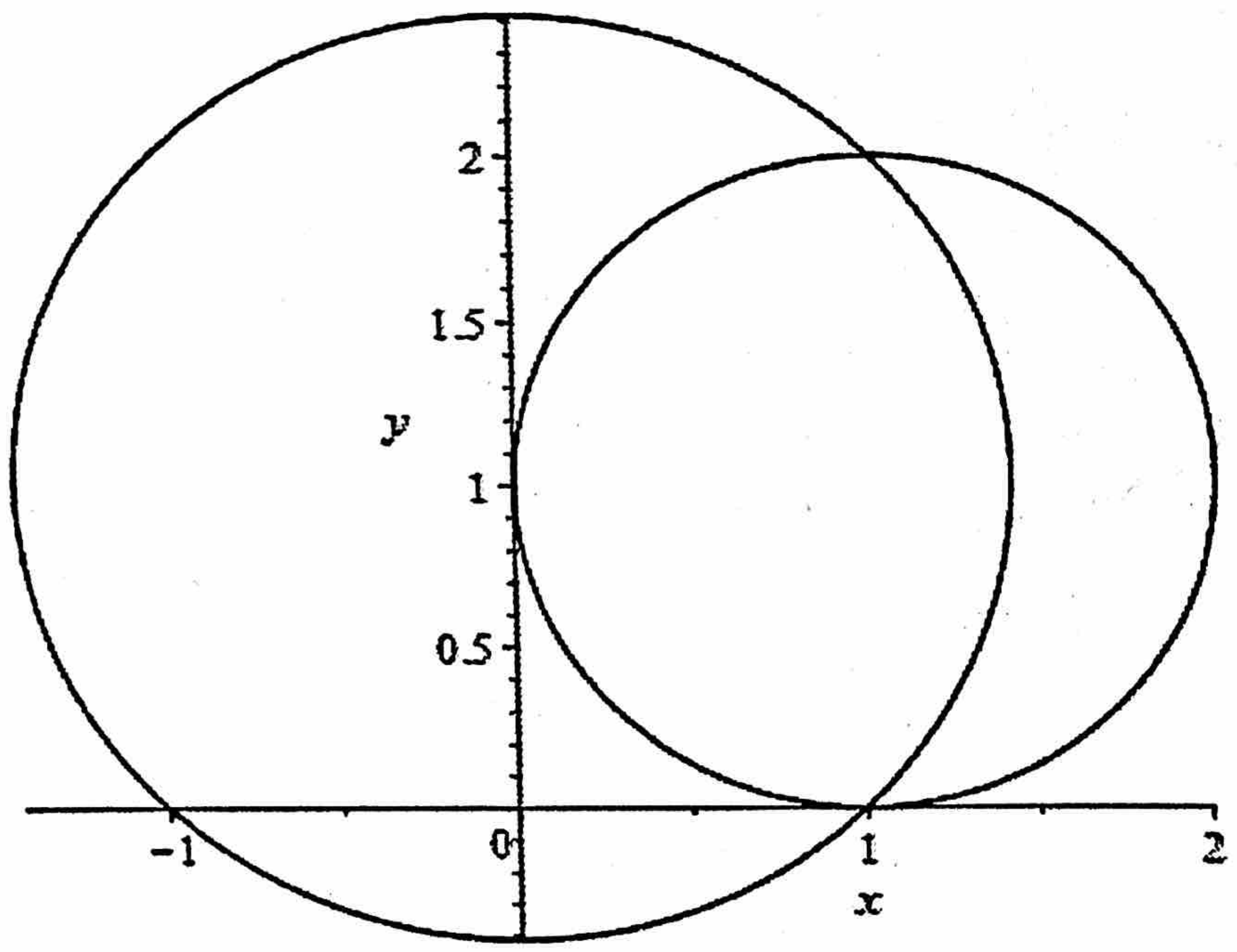
Exercise 4

By symmetry of rotation around O_z, we know that $G_x = G_y = 0$.

Since it is homogeneous, $G_z = \frac{1}{V}\iiint zdV$ where $V = \frac{2}{3}\pi R^3$ is the volume of the half ball.

Now, $\iiint zdV = \int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{R} r\cos(\theta)r^2\sin(\theta)\ dr\ d\theta\ d\varphi = 2\pi\frac{R^4}{4}\frac{1}{2} = \pi\frac{R^4}{4}$

So $G_z = \frac{3}{8}R$.



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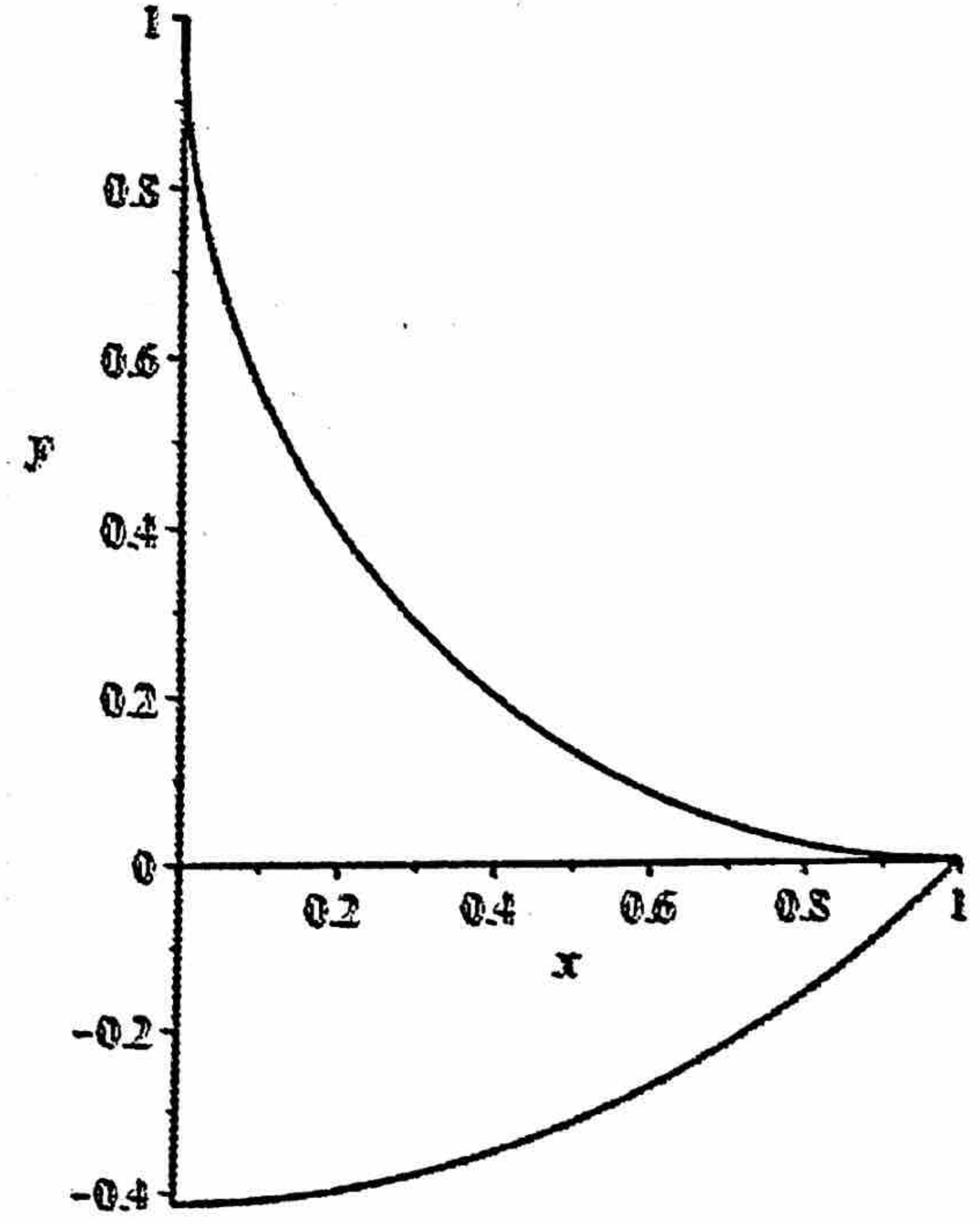
2
x



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0.2
0
0.2
0.4
0.6
0.8
1
x
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-0.4