INSTITUT NATIONAL DES SCIENCES APPLIQUÉES DE LYON
PÔLE DE MATHÉMATIQUES

**FIMI - SCAN1**

Test 2- MTES - 07/06/2019

**Warnings and advices**

- **Documents, dictionaries, phones, and calculators are FORBIDDEN.**
- **The marking scheme is only given as an indication.**

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## Exercise 1 *(7 pts)*

In $\mathbb{R}^2$, consider the points $A(-1,0)$, $B(1,0)$ and $C(0,1)$.

We denote by $\mathcal{P}$ the piece of parabola of equation $y = x^2 - 1$ between the points $A$ and $B$.

Finally, we denote $\Gamma$ the path $ABCA$ along $\mathcal{P}$ and the segments $[BC]$ and $[CA]$.

1. Represent $\Gamma$.
2. We consider the field $\overrightarrow{F_1}(x,y) = y^2\overrightarrow{e_x} + y\overrightarrow{e_y}$ defined on $\mathbb{R}^2$.
   (a) On a field map, represent the vector field at the 2 points $(0,1)$ and $(0,-1)$.
   (b) Show that $\overrightarrow{F_1}$ is not derived from a potential.
   (c) Compute the circulation of $\overrightarrow{F_1}$ along $\Gamma$.
   (d) Determine the equations of the field lines and represent the field line going through the point $(0,1)$ on the field map of question $2)a)$.
3. (a) Find a function $\alpha : \mathbb{R} \to \mathbb{R}$ depending only on the variable $x$ such that :
   - $\alpha(0) = 0$.
   - the vector field $\overrightarrow{F_2}(x,y) = y^2\overrightarrow{e_x} + y\alpha(x)\overrightarrow{e_y}$ is derived from a potential on $\mathbb{R}^2$.

   (b) Find the potential $P(x,y)$ from which $\overrightarrow{F_2}$ is derived such that $P(0,0) = 0$.
   (c) We consider the oriented half circle $\Gamma'$ centered at $O$ and going from the point $(-1,-1)$ to the point $(1,1)$. Compute the circulation of $\overrightarrow{F_2}$ on $\Gamma'$.
4. We denote by $S$ the flat surface inside $\Gamma$.
   (a) Give the normal vector to $S$ induced by the orientation of $\Gamma$.
   (b) Compute the flux of the field $\overrightarrow{F_3}(x,y,z) = x^2\overrightarrow{e_z}$ through $S$.

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**Exercise 2** *(6 pts)*

In $\mathbb{R}^3$ with the usual orthonormal frame $(O, \overrightarrow{e_x}, \overrightarrow{e_y}, \overrightarrow{e_z})$, we consider the vector field $\overrightarrow{F}$ defined on $\mathbb{R}^3$ minus the $Oz$ axis and given in spherical coordinates by $\overrightarrow{F}(r, \theta, \varphi) = \overrightarrow{e_r} + \pi r \sin(\theta)\overrightarrow{e_\varphi}$.

1. (a) Give the expression of the infinitesimal displacement $\overrightarrow{dOM}$ in spherical coordinates.
   (b) We are looking for the field lines as parametric curves in spherial coordinates : $(r(t), \theta(t), \varphi(t))$. Find the equations of the field lines and describe their trajectories in $\mathbb{R}^3$.
2. (a) Compute the infinitesimal circulation $\overrightarrow{F} \cdot \overrightarrow{dOM}$.
   (b) Show that the circulation of $\overrightarrow{F}$ on the circle of radius 1, centered at the origin and in the horizontal plane $z = 0$ is equal to $2\pi^2$.
   (c) Is the field derived from a potential ? If yes, give one possible potential.
3. We consider the surface $S$, defined in cartesian coordinates by

$$S = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1 \quad \text{and} \quad -1 \leqslant z\sqrt{2} \leqslant 1\}$$

   (a) Give a description of $S$ in spherical coordinates.
   (b) Compute the flux of the field $\overrightarrow{F}$ through the surface $S$.

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**Exercise 3** *(7 pts + 1 pt BONUS)*

We consider a surface of revolution $R_S$ defined in cylindrical coordinates by the equations

$$r^2 - z^2 = 9 \text{ and } z \in [0, 4]$$

The surface $R_S$ is naturally outwardly oriented.

1. Sketch the representation of a cut of the surface in the plane $y = 0$.
2. **BONUS :** Show that on $R_S$, $\overrightarrow{dS} = \left(\sqrt{9 + z^2}\overrightarrow{e_r} - z\overrightarrow{e_z}\right) \, \mathrm{d}\theta \, \mathrm{d}z$.
3. (a) Compute the flux of the vector field $\overrightarrow{F_1} = z\overrightarrow{e_z}$ through the surface $R_S$.
   (b) The surface $R_S$ is closed by 2 horizontal disks : the disk $D_0$ for $z = 0$ and the disk $D_4$ for $z = 4$ to obtain a surface $H$.
   Compute the total flux of $\overrightarrow{F_1}$ going out of $H$.
   (c) Compute the volume inside $H$ and check that the result is exactly the flux of the previous question.
4. Compute the flux of the vector field $\overrightarrow{F_2} = \dfrac{1}{\sqrt{9 + z^2}}\overrightarrow{e_x}$ through the surface $R_S$.

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