

TEST 2- MTES - 07/06/2019

Warnings and advices

- Documents, dictionaries, phones, and calculators are FORBIDDEN.
- The marking scheme is only given as an indication.

удачи

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EXERCISE 1 (7 pts)

In  $\mathbb{R}^2$ , consider the points  $A(-1, 0)$ ,  $B(1, 0)$  and  $C(0, 1)$ .

We denote by  $\mathcal{P}$  the piece of parabola of equation  $y = x^2 - 1$  between the points  $A$  and  $B$ .

Finally, we denote  $\Gamma$  the path  $ABCA$  along  $\mathcal{P}$  and the segments  $[BC]$  and  $[CA]$ .

1. Represent  $\Gamma$ .
  2. We consider the field  $\vec{F}_1(x, y) = y^2\vec{e}_x + y\vec{e}_y$  defined on  $\mathbb{R}^2$ .
    - (a) On a field map, represent the vector field at the 2 points  $(0, 1)$  and  $(0, -1)$ .
    - (b) Show that  $\vec{F}_1$  is not derived from a potential.
    - (c) Compute the circulation of  $\vec{F}_1$  along  $\Gamma$ .
    - (d) Determine the equations of the field lines and represent the field line going through the point  $(0, 1)$  on the field map of question 2)a).
  3. (a) Find a function  $\alpha : \mathbb{R} \rightarrow \mathbb{R}$  depending only on the variable  $x$  such that :
    - $\alpha(0) = 0$ .
    - the vector field  $\vec{F}_2(x, y) = y^2\vec{e}_x + y\alpha(x)\vec{e}_y$  is derived from a potential on  $\mathbb{R}^2$ .(b) Find the potential  $P(x, y)$  from which  $\vec{F}_2$  is derived such that  $P(0, 0) = 0$ .  
(c) We consider the oriented half circle  $\Gamma'$  centered at  $O$  and going from the point  $(-1, -1)$  to the point  $(1, 1)$ . Compute the circulation of  $\vec{F}_2$  on  $\Gamma'$ .
  4. We denote by  $S$  the flat surface inside  $\Gamma$ .
    - (a) Give the normal vector to  $S$  induced by the orientation of  $\Gamma$ .
    - (b) Compute the flux of the field  $\vec{F}_3(x, y, z) = x^2\vec{e}_z$  through  $S$ .
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**EXERCISE 2 (6 pts)**

In  $\mathbb{R}^3$  with the usual orthonormal frame  $(O, \vec{e}_x, \vec{e}_y, \vec{e}_z)$ , we consider the vector field  $\vec{F}$  defined on  $\mathbb{R}^3$  minus the  $Oz$  axis and given in spherical coordinates by  $\vec{F}(r, \theta, \varphi) = \vec{e}_r + \pi r \sin(\theta) \vec{e}_\varphi$ .

1. (a) Give the expression of the infinitesimal displacement  $\overrightarrow{dOM}$  in spherical coordinates.  
 (b) We are looking for the field lines as parametric curves in spherical coordinates :  $(r(t), \theta(t), \varphi(t))$ . Find the equations of the field lines and describe their trajectories in  $\mathbb{R}^3$ .
2. (a) Compute the infinitesimal circulation  $\vec{F} \cdot \overrightarrow{dOM}$ .  
 (b) Show that the circulation of  $\vec{F}$  on the circle of radius 1, centered at the origin and in the horizontal plane  $z = 0$  is equal to  $2\pi^2$ .  
 (c) Is the field derived from a potential? If yes, give one possible potential.
3. We consider the surface  $S$ , defined in cartesian coordinates by

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \quad \text{and} \quad -1 \leq z\sqrt{2} \leq 1\}$$

- (a) Give a description of  $S$  in spherical coordinates.
- (b) Compute the flux of the field  $\vec{F}$  through the surface  $S$ .

**EXERCISE 3 (7 pts + 1 pt BONUS)**

We consider a surface of revolution  $R_S$  defined in cylindrical coordinates by the equations

$$r^2 - z^2 = 9 \quad \text{and} \quad z \in [0, 4]$$

The surface  $R_S$  is naturally outwardly oriented.

1. Sketch the representation of a cut of the surface in the plane  $y = 0$ .
2. **BONUS** : Show that on  $R_S$ ,  $\overrightarrow{dS} = (\sqrt{9 + z^2} \vec{e}_r - z \vec{e}_z) \, d\theta \, dz$ .
3. (a) Compute the flux of the vector field  $\vec{F}_1 = z \vec{e}_z$  through the surface  $R_S$ .  
 (b) The surface  $R_S$  is closed by 2 horizontal disks : the disk  $D_0$  for  $z = 0$  and the disk  $D_4$  for  $z = 4$  to obtain a surface  $H$ .  
 Compute the total flux of  $\vec{F}_1$  going out of  $H$ .  
 (c) Compute the volume inside  $H$  and check that the result is exactly the flux of the previous question.
4. Compute the flux of the vector field  $\vec{F}_2 = \frac{1}{\sqrt{9 + z^2}} \vec{e}_x$  through the surface  $R_S$ .