Solutions and Grading Scheme for IE2S2
SCAN First 2020-2021 - 04/06/2021

Instructions: items in red are graded, items in black are for information only

EX1	**3 pts**
1.1	**Total: 1 pt**
$dC = \vec{B} \cdot d\overrightarrow{OM} = Pdx + Qdy + Rdz$	
$\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} = 0, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} = 0$ and $\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} = 0$	0,25 + 0,25 + 0,25
dC is hence **closed** and since P, Q and R are of class at least C^1 on $\mathbb{R}^3$ then it is also **exact**, then $\vec{B}$ is derived from a potential	0,25
1.2	**Total: 1,5pt**
$\vec{B} = -\nabla V$, accepted if $\vec{B} = \nabla V$ with the different answers that go with it	
$-\frac{\partial V}{\partial x} = y^2\cos(x) \Leftrightarrow v(x,y,z) = -y^2\sin(x) + f(y,z)$	0,5
$-\frac{\partial V}{\partial y} = -\frac{\partial}{\partial y}\left(-y^2\sin(x) + f(y,z)\right) = 2y\sin(x) + e^{2z} \Leftrightarrow f(y,z) = -ye^{2z} + g(z)$	0,5
$v(x,y,z) = -y^2\sin(x) - ye^{2z} + 1$ or $v(x,y,z) = y^2\sin(x) + ye^{2z} + 1$ (if $\vec{B} = \nabla V$)	0,5
1.3	**Total: 0,5pt**
$C = \int_x^y \vec{B} \cdot \vec{dl} = V(x) - V(y)$	0,25
$V(x) - V(y) = 0 - (-11) = 11$ or -11 (if $\vec{B} = \nabla V$)	0,25

EX2	**4 pts + 0,5 (bonus)**
2.1	**Total: 1 pt**
$\vec{F}(x,y) \times d\overrightarrow{OM} = \vec{0}$	
$\frac{x}{2}dy - ydx = 0$ and no variation in z	0,5
$y = k'x^2$	0,5
Parabolas in the xOy plane and oriented from the origin outwards (bonus)	0,5 + 0,5 (bonus)

2.2	**Total: 1pt**
$dC = \left[\frac{x}{2}\vec{e}_x + y\vec{e}_y\right] \cdot \left[dx\vec{e}_x + dy\vec{e}_y\right]$	
Closed, since $\frac{\partial y}{\partial x} = 0 = \frac{\partial \frac{x}{2}}{\partial y}$	0,25
$\frac{x}{2}$ and y are C^1 on $\mathbb{R}^2 \Rightarrow$ exact, hence $\vec{F}$ derives from a potential	0,25
For the potential $\frac{x}{2}\vec{e}_x + y\vec{e}_y = -\nabla\phi$	
Which gives $\Rightarrow \phi(x, y) = -\frac{x^2}{4} + f(y)$	0,25
The potential is $\phi(x, y) = -\frac{x^2}{4} - \frac{y^2}{2} + k$	0,25
2.3	**Total: 0,5pt**
$\phi(x, y) = C$ with C a constant	
$\frac{x^2}{4} + \frac{y^2}{4} = C'$ which is the equation of ellipses centered at the origin	0,5 + 0,5
appropriate sketch	0,5

EX3	**6 pts + 0,5 (bonus)**
3.1	**Total: 1 pt**
$\phi_{AOC} = \iint_{AOC} \vec{F} \cdot \vec{dS}$ with $y = 0$ and $\vec{dS} = (-\vec{e}_y)\ dx dz$	
$\phi_{AOC} = 0$, and similarly $\phi_{COB} = 0$ and $\phi_{AOB} = 0$	0,25 + 0,25 + 0,25
They are the same since $\vec{F}$ is always normal to $\vec{dS}$	0,25

3.2	**Total: 2,75pt**
One possible parametrisation is by using the vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$:	
$\overrightarrow{OM} = u\overrightarrow{CA} + v\overrightarrow{CB} + 2\vec{e}_z$	
$\overrightarrow{OM} = u\left(\vec{e}_x - 2\vec{e}_z\right) + v\left(\vec{e}_y - 2\vec{e}_z\right) + 2\vec{e}_z = u\vec{e}_x + v\vec{e}_y + 2(-u - v + 1)\vec{e}_z$, $u, v \in [0,1]$	1
Let $\vec{p} = \dfrac{\partial \overrightarrow{OM}}{\partial u} \times \dfrac{\partial \overrightarrow{OM}}{\partial v}$ be a vector normal to ABC	0,5
$\dfrac{\partial}{\partial u}\left[u\vec{e}_x + v\vec{e}_y + 2(-u - v + 1)\vec{e}_z\right] = \vec{e}_x - 2\vec{e}_z$	0,25
$\dfrac{\partial}{\partial v}\left[u\vec{e}_x + v\vec{e}_y + 2(-u - v + 1)\vec{e}_z\right] = \vec{e}_y - 2\vec{e}_z$	0,25
And hence, $\vec{p} = 2\vec{e}_x + 2\vec{e}_y + \vec{e}_z$	0,5
Finally $\vec{n} = \dfrac{\vec{p}}{\lvert\lvert \vec{p} \rvert\rvert} = \dfrac{1}{3}\left[2\vec{e}_x + 2\vec{e}_y + \vec{e}_z\right]$, cqfd	0,25
3.3	**Total: 0,5pt (bonus)**
We already have a parametrisation as seen in 3.2. We need to demonstrate that it is equivalent to the one given.	
If we take $u = ab$ and $-u - v = -a$ then we have that $v = a(1-b)$ we can rewrite the parametrisation of 3.2 replacing u and v and get $\overrightarrow{OM} = ab\vec{e}_x + a(1-b)\vec{e}_y + 2(1-a)\vec{e}_z$ with. $a, b \in [0,1]$	0,5 (bonus)
3.4	**Total: 1pt**
$\phi_{ABC} = \iint_{ABC} \vec{F}_{uv} \cdot d\vec{S}_{uv}$ with $d\vec{S} = \dfrac{1}{3}\left[2\vec{e}_x + 2\vec{e}_y + \vec{e}_z\right]\, du\, dv$ and $u, v \in [0,1]$	0,25
$= \displaystyle\int_{u=0}^{1}\int_{v=0}^{1} \left\{u\vec{e}_x + v\vec{e}_y + \frac{1}{2}[2(-u - v + 1)]\,\vec{e}_z\right\} \cdot \left\{\frac{2}{3}\vec{e}_x + \frac{2}{3}\vec{e}_y + \frac{1}{3}\vec{e}_z\right\} du\, dv$	0,25
$= \displaystyle\int_0^1 \left[\frac{1}{3}uv + \frac{v^2}{6} + \frac{1}{3}v\right]_0^1 du$	0,25
$= \dfrac{1}{2}$	0,25

3.5	Total: 0,75pt
$C_{AO} = \int_{AO} \vec{F} \cdot d\overrightarrow{OM}$ with $AO: y = 0, z = 0, x \in [0,1]$ and $d\overrightarrow{OM} = dx\, \vec{e}_x$	
$C_{AO} = \frac{1}{2}$	0,25
$C_{OB} = \int_{OB} \vec{F} \cdot d\overrightarrow{OM}$ with $OB: y \in [0,1], z = 0, x = 0$ and $d\overrightarrow{OM} = dy\, \vec{e}_y$	
$C_{AO} = -\frac{1}{2}$	0,25
Not the same due to the symmetry of $\vec{F}$ on x and y $\Rightarrow C_{AO} = C_{BO}$ and hence $\Rightarrow C_{AO} = -C_{OB}$	0,25
3.6	Total: 0,5pt
$\vec{F}$ is conservative since it is derived from a potential	
$\frac{\partial}{\partial x}\left(\frac{z}{2}\right) = 0 = \frac{\partial}{\partial z}(x)$, $\frac{\partial}{\partial x}(y) = 0 = \frac{\partial}{\partial y}(x)$ and $\frac{\partial}{\partial y}\left(\frac{z}{2}\right) = 0 = \frac{\partial}{\partial z}(y)$, hence closed	0,25
$\vec{F}$ is C^1 on $\mathbb{R}^3 \Rightarrow$ exact, hence $\vec{F}$ derives from a potential $\Rightarrow C = 0$ on a closed path	0,25

EX4	4 pts
4.1	Total: 1 pt
$\phi_{D_H} = \iint_{D_H} \vec{E}_1 \cdot \overrightarrow{dS}$ with $\overrightarrow{dS} = r d\theta dr \vec{e}_z$	
Since $\vec{E}_1$ is according to $\vec{e}_r$, then $\vec{E}_1 \cdot \overrightarrow{dS} = 0$ and hence $\phi_{D_H} = 0$	0,25
$\phi_{D\_H} = \iint_{D\_H} \vec{E}_1 \cdot \overrightarrow{dS}$ with $\overrightarrow{dS} = r d\theta dr(-\vec{e}_z)$	
same thing for $\phi_{D\_H} = 0$	0,25
4.2	Total: 1pt
$\phi_C = \iint_C \vec{E}_1 \cdot \overrightarrow{dS}$ with $\overrightarrow{dS} = r d\theta dz \vec{e}_r$ and $r = a$	0,5
$\phi_C = 4\pi a H E(a)$	0,5
4.3	Total: 0,5pt
$\phi_A = \phi_{D_H} + \phi_{D\_H} + \phi_C$ and given the prior results we have	
$\phi_A = 0 + 0 + 4\pi a H \frac{4a}{1+a^2} = \frac{16\pi H a^2}{1+a^2}$	0,5
A.N. $\phi_A = \frac{64\pi}{5}$	0,5

4.4	**Total: 1,5pt**
$C_{AB} = \int_B^A \vec{E} \cdot \overrightarrow{dOM}$ with $\overrightarrow{dOM} = dr\vec{e}_r + rd\theta\vec{e}_\theta + dz\vec{e}_z$	0,5
With the variable change $u = 1 + r^2$; $du = 2rdr$ we have $C_{AB} = 2\int_1^5 \frac{1}{u}du = 2\left[\ln(u)\right]_1^5$	0,5
$C_{AB} = 2ln(5)$	0,5

EX5	**3 pts**
$C_{ABCOA} = C_{AB} + C_{BO} + C_{OC} + C_{CA}$	
$C_{AB} = \int_{AB} \vec{F} \cdot \overrightarrow{dOM}$ with $r = 1, \phi = 0, \theta = \pi/2 \rightarrow 0$ and $\overrightarrow{dOM} = rd\theta\vec{e}_\theta = d\theta\vec{e}_\theta$	0,5
$C_{AB} = -1$	0,25
$C_{BO} = \int_{BO} \vec{F} \cdot \overrightarrow{dOM}$ with $r = 1 \rightarrow 0, \phi =$ undefined (not a problem since only $\vec{e}_r$ remains), $\theta = 0$ and $\overrightarrow{dOM} = dr\vec{e}_r$	0,5
$C_{AB} = -\pi/4$	0,25
$C_{OC} = \int_{OC} \vec{F} \cdot \overrightarrow{dOM}$ with $r = 0 \rightarrow 1, \phi = 0, \theta = \pi/2$ and $\overrightarrow{dOM} = dr\vec{e}_r$	0,5
$C_{AB} = \pi/2$	0,25
$C_{CA} = \int_{CA} \vec{F} \cdot \overrightarrow{dOM}$ with $r = 1, \phi = \pi/2 \rightarrow 0, \theta = \pi/2$ and $\overrightarrow{dOM} = r\sin\theta d\phi\vec{e}_\phi = d\phi\vec{e}_\phi$	0,25
$C_{AB} = -\pi^3/16$	0,25
finally $C_{ABCOA} = \frac{-\pi^3 + 4\pi - 16}{16}$	0,25