INSTITUT NATIONAL DES SCIENCES APPLIQUÉES DE LYON

FIMI - SCAN - First

IE Fields MTES - Friday, June 4th, 2021 - Duration 2h

Instructions :

- Calculators, documents and mobile phones are **not allowed**
- A minimum of calculations or justifications are expected for all exercises !
- The presentation, the quality of writing, and the explicit reasoning will be taken into account when grading.

Good luck !

Exercise 1 ***(3 pts)***

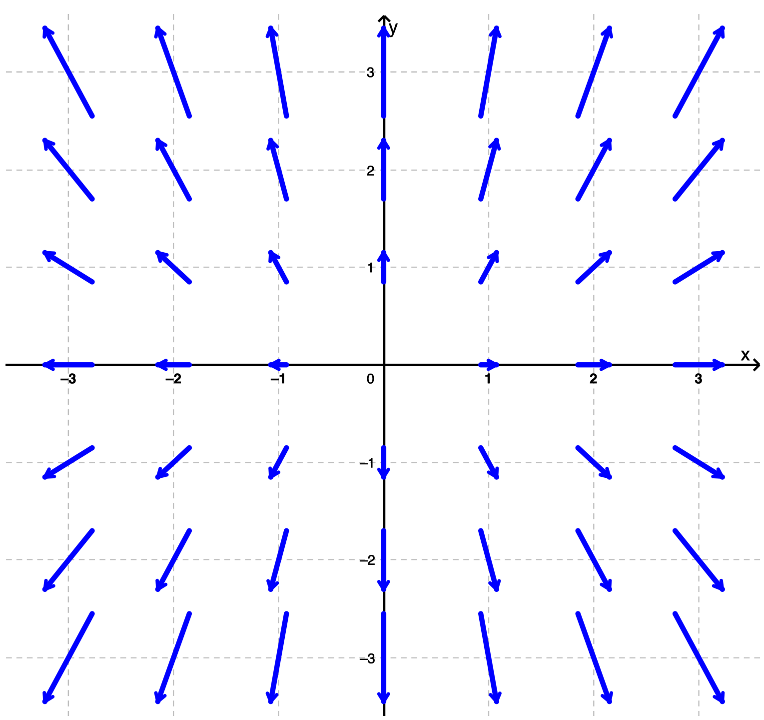
Consider a vector field given by $\overrightarrow{B}(x, y, z) = y^2 \cos(x)\overrightarrow{e}_x + \left(2y\sin(x) + \mathrm{e}^{2z}\right)\overrightarrow{e}_y + 2y\mathrm{e}^{2z}\overrightarrow{e}_z$

1. Calculate the elementary circulation $dC = \overrightarrow{B} \cdot d\overrightarrow{OM}$ and show that it is closed. What can we deduce from this ?
2. Determine the potential $V(x, y, z)$ from which this vector field derives, knowing that $V = 1$ at the origin.
3. What is the circulation of $\overrightarrow{B}$ from point $X(0, 1, 0)$ to point $Y(\pi/2, 3, 0)$?

Exercise 2 ***(4 pts)***

Let $\overrightarrow{F}(x, y) = \dfrac{x}{2}\overrightarrow{e}_x + y\overrightarrow{e}_y$ be a vector field in $\mathbb{R}^2$ and represented by the figure below.

1. Determine the field lines' expression. On a graph, sketch as precisely as possible **one** field line obtained. **Bonus :** express the orientation of the field line.
2. Show that $\overrightarrow{F}$ derives from a potential.
3. Determine the equipotentials' expression. On **the same graph as for Q.1**, sketch as precisely as possible **one** equipotential obtained.



Exercise 3 *(6 pts)*
Let $\overrightarrow{F}(x,y,z) = x\overrightarrow{e}_x + y\overrightarrow{e}_y + \frac{z}{2}\overrightarrow{e}_z$ be a vector field defined on $\mathbb{R}^3$. In a cartesian frame and coordinates let the points $A(1,0,0)$, $B(0,1,0)$, $O(0,0,0)$ and $C(0,0,2)$, define the boundary of a prism, **with outward orientation**.

1. Compute the fluxes Φ_{AOC}, Φ_{COB} and Φ_{AOB} of $\overrightarrow{F}$ through the lateral surfaces of the prism, respectively the triangles AOC, COB and AOB. Are they the same? If yes, why? If not, why aren't they the same?
2. Show that the oriented surface of the triangle ABC accepts $\left[\frac{2}{3}\overrightarrow{e}_x + \frac{2}{3}\overrightarrow{e}_y + \frac{1}{3}\overrightarrow{e}_y\right]$ as a normal vector.
3. Show that $[uv\overrightarrow{e}_x + u(1-v)\overrightarrow{e}_y + 2(1-u))\overrightarrow{e}_z]$, with $0 \leqslant u, v \leqslant 1$ is a possible parametrisation of the the triangle ABC.
4. Compute the flux Φ_{ABC} of $\overrightarrow{F}$ through the surface of the triangle ABC.
5. Calculate the circulations C_{AO} and C_{OB} of $\overrightarrow{F}$ on the segments $[A,O]$ and $[O,B]$. Are they the same? If yes why? If not, why aren't they the same?
6. Calculate the circulation of $\overrightarrow{F}$ on the closed path $ABCA$.

Exercise 4 *(4 pts)*
Consider a space in $\mathbb{R}^3$ given by cylindrical coordinates (r,θ,z). A device creates an electrical field $\overrightarrow{E}_1 = E(r)\overrightarrow{e}_r$, equal to zero on the (Oz) axis ($E(0) = 0$). Let C be a **closed** cylinder of dimensions : $-H \leqslant z \leqslant H$ and $0 \leqslant r \leqslant a$. The cylinder is bounded at the top by a disk D_H and at the bottom by another disk D_{-H}, as well as around its axis at $r = a$.

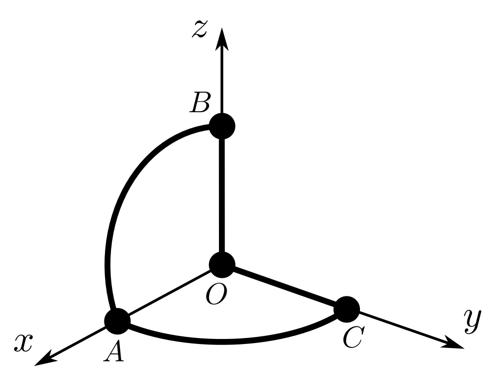
1. Show that the flux of $\overrightarrow{E}_1$ exiting the disks D_H and D_{-H} delimiting C are both equal to zero.
2. Express the flux of $\overrightarrow{E}_1$ exiting the lateral surface of the cylinder, as a function of a, H and $E(a)$.

Now let us consider a special case of the field described above, where $E(r) = \dfrac{4r}{1+r^2}$.

3. Calculate the total flux of $\overrightarrow{E}_1$ through **all sides** of the cylinder C, for $a = 2$ and $H = 1$.
4. Calculate the circulation of $\overrightarrow{E}_1$ on the line segment [A,B], with A(0,0,1) and B(2,0,1).

Exercise 5 *(3 pts)*
The set of points A(1,0,0), B(0,0,1), C(0,1,0) and the origin define the connecting points of the following paths (as seen on the figure) :

- AB : 1/4 of a circle in the xOz plane with radius 1, centered at the origin;
- BO : line on the Oz axis with length 1;
- OC : line on the Oy axis with length 1;
- CA : 1/4 of a circle in the xOy plane with radius 1, centered at the origin.



Calculate the circulation of the field $\overrightarrow{F}(r,\theta,\phi) = (\theta - \frac{\pi}{4})\overrightarrow{e}_r + \sin(\theta)\overrightarrow{e}_\theta + \theta\phi\overrightarrow{e}_\phi$ (given in spherical coordinates and frame) on the closed boundary path ABOCA. Remember that θ is the angle starting from the Oz axis and ϕ the angle in the xOy plane, in spherical coordinates.

Good luck!