Solutions and Grading Scheme for IE1S2
SCAN First 2021-2022 - 06/05/2022

Instructions: items in **black** are graded, items in **gray** are for information only

EX1	**4,5 pts**
1.1 I $\Longrightarrow$ C since the derivative of $y = \sqrt{ax}$ at 0 tends to infinity $\Longrightarrow$ vertical tangent at O J $\Longrightarrow$ A since the derivative of $y = \frac{x^2}{a}$ at 0 tends to 0 $\Longrightarrow$ horizontal tangent at O K $\Longrightarrow$ B trivial since it is the line $y = x$	**Total: 0,5 pt or 0**
1.2 $I' = \int_{y=0}^{a}\int_{x=0}^{\frac{y^2}{a}} dx\,dy$ $J' = \int_{y=0}^{a}\int_{x=0}^{\sqrt{ax}} dx\,dy$ $K' = \int_{y=0}^{a}\int_{x=0}^{y} dx\,dy$	**Total: 1,5pt** 0,5 : 0,5 : 0,5
1.3 $I = \int_0^a (a - \sqrt{ax})\,dx = \left[ax - \frac{2}{3a}(ax)^{\frac{3}{2}}\right]_0^a = \frac{a^2}{3}$ or as normal in y: $I' = \int_{y=0}^{a} \frac{y^2}{a}\,dy = \frac{1}{3}[y^3]_0^a = \frac{a^3}{3}$ $J = \int_{x=0}^{a} \left(a - \frac{x^2}{a}\right) dx = \left[ax - \frac{x^3}{3a}\right]_0^a = \frac{2a^2}{3}$ or as normal in y: $J' = \int_{y=0}^{a} \sqrt{ay}\,dy = \left[\frac{2}{3a}(ay)^{\frac{3}{2}}\right]_0^a = \frac{2a^2}{3}$ $K = \int_{x=0}^{a} (a - x)\,dx = \left[ax - \frac{x^2}{2}\right]_0^a = \frac{a^2}{2}$ or as normal in y: $K' = \int_{y=0}^{a} y\,dy = \frac{a^2}{2}$	**Total: 2,5pt**

EX2	**6pts+1bonus**
2.1 Mass of the cylinder: $m_c = \rho\pi R^2 H$~3.1g (π g tolerated) Mass of the cone: $m_\Delta = \rho\frac{\pi R'^2 H'}{3}$~50g ($16\pi$ g tolerated)	**Total: 1pt** 0,5 0,5 (0,25 instead of 0,5 if no units)
2.2 Oz is the axis of revolution of the cylinder (ρ does not depend on θ), so G_C is on Oz and the perpendicular plane cutting the cylinder in two halves is a symmetry plane (ρ does not depend on the height) so G_C is ai the middle of the cylinder: $G_C\left(0{,}0, H' + H/2\right)$	**Total: 1pt** 0,5+0,5 (0 if not justified)
2.3 Oz is the axis of revolution of the cone, so G_Δ is on Oz: $G_\Delta(0{,}0, z_\Delta)$ In cylindrical coordinates: $r_\Delta = 0$, $\theta_\Delta = 0$ leaving us with only z_Δ to be determined. We have $dM = \rho dV = \rho r\,dr\,d\theta\,dz$ The variables r and z are connected through the expression: $\frac{R'}{H'} = \frac{r}{z}$, hence we can express, for example, r as a function of z $M_\Delta z_\Delta = \int_{z=0}^{H'}\int_{r=0}^{\frac{R'}{H'}z}\int_{\theta=0}^{2\pi} z \cdot \rho r\,dr\,d\theta\,dz$ $= 2\pi\rho\int_{z=0}^{H'}\int_{r=0}^{\frac{R'}{H'}z} z \cdot r\,dr\,dz = \pi\rho\int_{z=0}^{H'}[r^2]_{r=0}^{\frac{R'}{H'}z} z\,dz$ $= \pi\rho\frac{R'^2}{H'^2}\int_{z=0}^{H'} z^3\,dz = \pi\rho\frac{R'^2}{4H'^2}H'^4 = \pi\rho\frac{R'^2 H'^2}{4}$ With the mass of the cone found in 2.1 we have: $z_\Delta = \frac{3}{4}H'$~2.2cm Bonus: $H' > z_\Delta$ > H'/2 which makes sense !	**Total: 3pt + 0,5 bonus** -not counted if already counted in the previous question (0,5 if point not given above) 0,5 1 (pose the integral) 1 (resolution) 0,5 (solution) Bonus 0,5 for the comment (in literal or numerical)

2.4	Total: 1pt + 0,5 bonus
For the overall center of gravity we will use the formula of the barycenter: $(M_\Delta + M_C)\vec{OG} = M_C\vec{OG}_C + M_\Delta\vec{OG}_\Delta$	0,25
from where we take the z-coordinate:	
$(M_\Delta + M_C)z_G = M_C z_C + M_\Delta z_\Delta \Longrightarrow z_G = \frac{M_C z_C + M_\Delta z_\Delta}{M_\Delta + M_C}$	0,5
N.A. $z_G = \frac{79}{34} cm$ ~2,3cm	0,5 (no units no points
Bonus : Mc<<M_Δ so $z_G \sim z_\Delta$ (or if the previous bonus was not given: any comment on the fact that the value 2,3cm "looks nice" or "makes sense")	Bonus 0,5

EX3	5,5 pts +0,5 bonus
3.1	**Total: 2 pt + 0,5 bonus**
Mass: $M = \iiint_V dM = \iiint_V \rho dV = \iiint_V k\frac{a^2+r^2}{4a+z} r\, d\theta\, dr\, dz$	0,5 (expression)
The bounds of the integral are independent and the function to integrate can be expressed as a product of type $f(r)\cdot g(\theta)\cdot h(z)$ hence we can apply Fubini, writing it as a product of integrals:	
$M = k\int_0^{2\pi} d\theta \cdot \int_0^a r(a^2+r^2)dr \cdot \int_{-2a}^{2a} \frac{1}{4a+z} dz$	
$M = k\cdot 2\pi \cdot \left[\frac{a^2}{2}r^2 + \frac{1}{4}r^4\right]_0^a \cdot [ln(4a+z)]_{-2a}^{2a}$	0,5 (for posing correctly the integral with the right bounds)
$M = 2\pi k\left(\frac{a^4}{2}+\frac{a^4}{4}\right) ln(3)$	
$M = \pi k\frac{3a^4}{2} ln(3)$	1 (calculation and result)
Bonus : dimensions are correct (no dimension in the ln and [k]=ML^{-4})	Bonus 0,5
3.2	**Total: 1pt**
Parametrization of the cylinder in Cartesian coordinates:	
$C = \{M(x,y,z) \in \mathbb{R}^3 \mid (x^2+y^2) \le a^2 \ and -2a \le z \le 2a\}$ (or any equivalent form)	1
3.3	**Total: 1pt**
sketch	0,5
Distance between the point $M(x,y,z)$ and the axis Oy: $r_{Oy} = \sqrt{z^2+x^2}$	0,5

3.4 Moment w.r.t. Oy: $J = \iiint_V r_{Oy}^2 dM = \iiint_V r_{Oy}^2 \rho dV = \iiint_V (x^2+z^2) k \frac{a^2+(x^2+z^2)}{4a+z} dx dy dz$ With bounds: $J = k \int_{z=-2a}^{2a} \int_{y=-1}^{1} \int_{x=-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (x^2+z^2) \frac{a^2+(x^2+z^2)}{4a+z} dx dy dz$	**Total: 1,5pt** 0,5 (for the moment of inertia correctly posed with r_oy and rhodV) 1 (complete integral with bounds)

Exercise 4	**4 pts+0,5 bonus**
4.1 $\vec{0} = \frac{M_S}{M_S+M}\vec{Ob} + \frac{M}{M_S+M}\vec{Oa} \Rightarrow 0 = -\frac{M_S}{M_S+M} b + \frac{M}{M_S+M} a$ or any equivalent equation It follows that: $M_S = M\frac{a}{b}$	**Total: 1 pt** 0,50 0,50
4.2 The height of the liquid in the tank has **cylindrical symmetry**, hence, we should use **cylindrical coordinates** $V_S = \iiint_{(S)} dV$, with the parametrisation of the sphere as such: $\theta \in [0,2\pi]$, $z \in [-R, h-R]$ (with the reference at the center of the sphere, for a much simpler calculation), $r \in [0, \sqrt{R^2-z^2}]$, and $dV = r d\theta dr dz$ Hence: $V_S = \int_{\theta=0}^{2\pi} \int_{z=-R}^{h-R} \int_{r=0}^{\sqrt{R^2-z^2}} r dr dz d\theta$, which after resolution should give: $V_S = \pi\left(h^2R - \frac{h^3}{3} + \frac{2}{3}R^3\right)$	**Total: 2,5pt** 0,25 (0 if other conclusion, even if the integral and the result is correct in the end) 4 * 0,25 (For this or any similar <u>good parametrization</u>, and dV) 1,25
4.3 $M_S = \rho V_S = \rho\pi\left(R^2h - \frac{h^3}{3} + \frac{2}{3}R^3\right)$ Then, from the result in 4.1 it follows that: $\frac{\rho\pi b}{M}\left(R^2h - \frac{h^3}{3} + \frac{2}{3}R^3\right) = a$ NB: this is the equation the engineer should program in its computer that controls the pump Bonus: check dimensions either in Q4.2 or Q4.3	**Total: 0,5 pt + 0,5 bonus** 0,25 0,25 Bonus 0,5