Physics test 3 - Friday 21st 2019

Exercise 1: Electric Locomotive (7.0 pts)

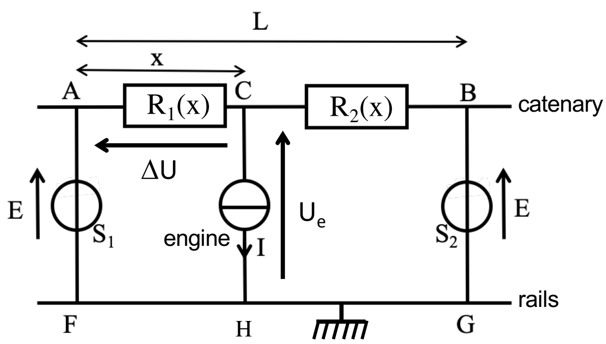
1. Given the linear resistance r of the catenary and the distances between the locomotive and the sub-stations ($AC = x$ and $CB = L - x$): **1.0 pt**

$$R_1 = r \cdot AC = r \cdot x$$ (0.5 pt)

$$R_2 = r \cdot CB = r \cdot (L - x)$$ (0.5 pt)

2. **3.0 pts**

Let I_1 be the current flowing in AC branch, the scheme of the circuit is depicted below:



defining I_1+scheme (0.5 pt) Note: 2 currents can defined (in AC and CB) but Kirchhoff's current law has to be used later during the resolution

Given the definition of I_1, ΔU is given by Ohm's law : $\Delta U = R_1 \cdot I_1$

$$\Delta U = r\,x \cdot I_1$$ (0.5 pt)

I_1 has to be determined to find the expression of ΔU. We will then solve the circuit using Kirchhoff's circuit laws.

Using Kirchhoff voltage law in loop $ACHF$:

$$E - R_1 I_1 - U_e = 0$$

hence:

$$E - rxI_1 - U_e = 0 \quad (1)$$

Statement of KVL (0.25 pt) + expression (0.25 pt)

Using Kirchhoff voltage law in loop $CBGH$:

$$U_e - R_2(I_1 - I) - E = 0$$

hence:

$$U_e - r(L - x) \cdot (I_1 - I) - E = 0 \quad (2)$$

Statement of KVL (0.25 pt) + expression (0.25 pt)

(1)+(2) leads to:

$$-I_1 \cdot rL + r(L - x) \cdot I = 0$$

$$I_1 = \frac{L - x}{L} I$$

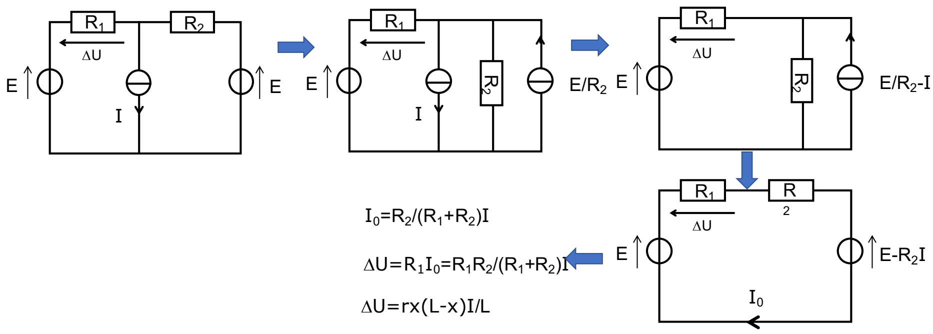
(0.5 pt) or any equiv. (see Note in Q2)

Finally:

$$\Delta U = \frac{x(L - x)}{L} rI$$

(0.5 pt)

<u>Note:</u> Given the locomotive is in between nodes A & B (resp. F & G), there is no current flow before A and after B (resp. before F and after G). Under these conditions, ABGF can be consdered as an isolated loop. It is therefore possible to simplify this circuit using successive transformation as shown below:



3.

We have: **2.0 pts**

$$\Delta U = \frac{x(L-x)}{L} rI$$

To find the position for which ΔU is maximum, let's find the position for which $\frac{d\Delta U}{dx} = 0$:

$$\frac{d\Delta U}{dx} = \frac{L-2x}{L} rI$$ (0.5 pt)

As $x = L/2$, $\frac{d\Delta U}{dx} = 0$. (0.25 pt)

Moreover: (0.25 pt) or any other justification

$$\frac{d^2\Delta U}{dx^2}(x = L/2) = -\frac{2rI}{L} < 0$$

so that $x = L/2$ corresponds to a maximum value of ΔU.

The maximum voltage drop is therefore:

$$\Delta U_{max} = \frac{rLI}{4}$$ (0.5 pt)

As $x = L/2$ the train is exactly in between S_1 and S_2. In this situation, the minimum resistance between a sub-station and the engine ($min(R_1, R_2)$) reaches a maximum value ($R_1(L/2) = R_2(L/2) = xrL/2$) so that a maximum voltage drop between S_1 or S_2 and the engine is reached. (0.5 pt)

4. **1.0 pt**

In order to keep $\Delta U_{max} \leq \Delta U_{crit}$ we should have:

$$\frac{rLI}{4} \leq \Delta U_{crit}$$

hence:

$$Lmax = \frac{4\Delta U_{crit}}{rI}$$ (0.5 pt)

We find: $Lmax = 4.5\,\mathrm{km}$ (0.5 pt)

Exercise 2: RL circuit (5.0 pts + 0.5 pts (bonus))

Step 1: Analyzing the initial conditions

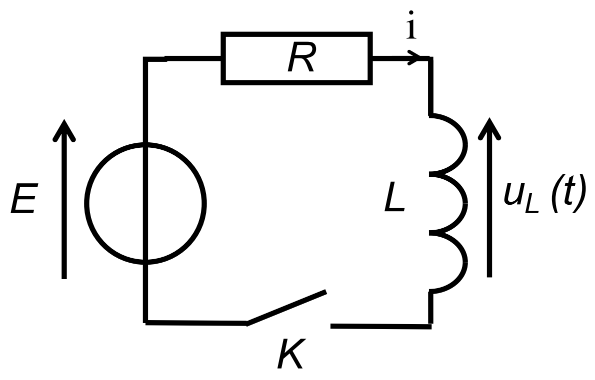
Let i be the current flowing in the circuit: **1.0 pt + 0.25 pt (bonus)**

Since K was opened for a sufficiently long time, we can assume that the current was nil in the circuit:

$$i(t < 0) = 0$$ (0.25 pt)

As the switch is closed, the current will increase after a transient state. Given the presence of an inductor, i is continuous so that:

$$i(t = 0^+) = i(t = 0^-) = i(t = 0) = 0$$ (0.5 pt) i(0)+justification



scheme + defining i (0.25 pt)

In addition, a steady-state was achieved before K was opened so that:

+(0.25 pt)

$$u_L(t<0) = L\frac{di}{dt}(t<0) = 0$$

In addition,

Step 2: Finding the circuit's equation $\boxed{\textbf{1.5 pt + 0.25 pt (bonus)}}$

Using Kirchoff's voltage law in the circuit for $t > 0$:

$$u_L + Ri = E \tag{3}$$ (0.5 pt)

Given $u_L = L\frac{di}{dt}$: (0.5 pt) for $u_L = \dots$

$$L\frac{di}{dt} + Ri = E$$

hence:

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}$$ (0.5 pt)

Let $\tau = L/R$ the circuit's equation writes:

$$\frac{di}{dt} + \frac{1}{\tau}i = \frac{E}{L} \tag{4}$$ bonus (+0.25 pt)

Note: we may also solve the circuit by differentiating (3) over time:

$$\frac{du_L}{dt} + R\frac{di}{dt} = \frac{du_L}{dt} + \frac{R}{L}u_L = 0$$

Step 3: Solving the equation $\boxed{\textbf{2.5 pts}}$

The solution writes:

$$i(t) = i_{gs}(t) + i_{ps}(t)$$

where i_{gs} is a general solution and i_{ps} a particular solution.

i_{gs} is a solution of:

$$\frac{di_{gs}}{dt} + \frac{1}{\tau}i_{gs} = 0$$

hence:

$$i_{gs}(t) = I_0 e^{-t/\tau}, \quad I_0 \in \mathbb{R}$$ (0.5 pt)

A particular solution can be found looking at the steady-state regime ($\frac{di}{dt} = 0$), (4) rewrites:

$$\frac{1}{\tau}i_{ps} = \frac{E}{L}$$

hence:

$$i_{ps}(t) = \tau E/L = E/R$$ (0.5 pt)

Finally:

$$i(t>0) = I_0 e^{-t/\tau} + \frac{E}{R}, \quad I_0 \in \mathbb{R}$$

Note: At this stage, we can either write $u_L = L\frac{di}{dt}$ to find u_L or fully determine $i(t)$ using the initial conditions. In both cases, I_0 has to be determined using the initial conditions. In the following, the second method is developed.

I_0 can be found using initial conditions:

$$i(0^+) = i(0^-) = i(0) = 0 \Rightarrow I_0 = -\frac{E}{R}$$ (0.25 pt)

so that:

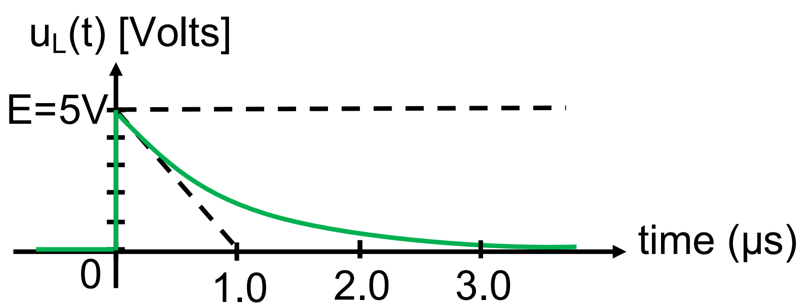
$$i(t \geq 0) = \frac{E}{R}\left(1 - e^{-t/\tau}\right)$$

and:

$$u_L(t \geq 0) = L\frac{di}{dt}(t \geq 0) = Ee^{-t/\tau}$$ (0.5 pt)

numerical value: $\tau = L/R = 1\,\mu\text{s}$ (0.25 pt)

Plot of $u_L(t)$:



plot ($t<0$ and $t>0$)+scales (0.5 pt)

Exercise 3: Neon Lamp (8.0 pts + 1.5 pts (bonus))

1. **1.5 pts**

U_{ON} and U_{OFF} can be easily found from the IV characteristic of the neon lamp:

$$U_{ON} = 10\,\text{V} \quad \text{and} \quad U_{OFF} = 50\,\text{V}$$ (0.5 pt)

To determine r_e we exploit the IV characteristic of the lamp as it is turned ON, using the uncertainty boxes.

Given the lamp is supposed to be equivalent to a resistor, it's IV characteristic should be a straight line passing through (0V,0A) with: (0.25 pt)

$$i = U/r_e$$

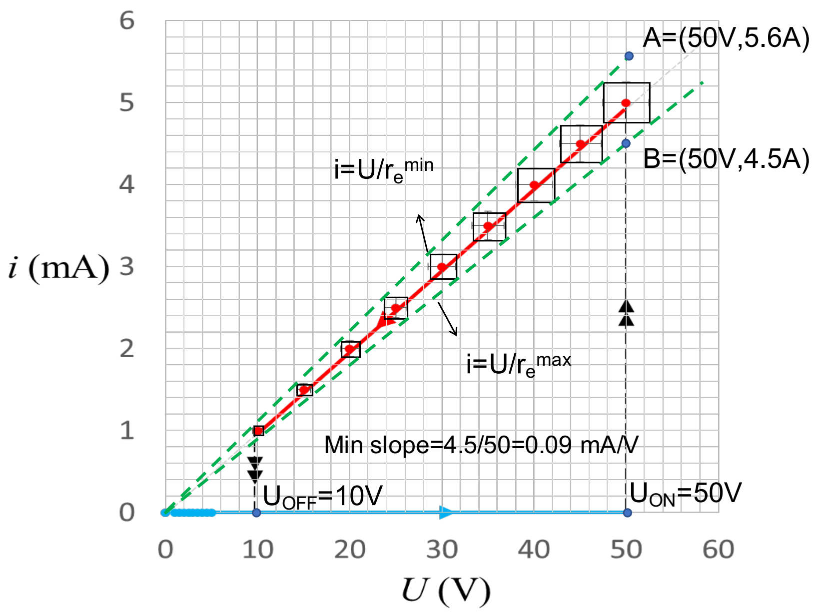
Using two points $A = (50\,\text{V}, 5.6\,\text{A})$ and $A = (50\,\text{V}, 4.5\,\text{A})$ located on the lines of max/min slopes, we can find r_e together with it's uncertainty:

$$r_e^{min} = \frac{1}{slope^{max}} \simeq \frac{50}{5.6\,10^{-3}} \simeq 8.929\,\text{k}\Omega$$

$$r_e^{max} = \frac{1}{slope^{min}} \simeq \frac{50}{4.5\,10^{-3}} \simeq 11.111\,\text{k}\Omega$$ (0.25+0.25 pt)

so that:

$$r_e^{mean} = \frac{r_e^{min} + r_e^{max}}{2} \simeq 10.02\,\text{k}\Omega$$



(-0.25 pts) if lines are not passing through (0V,0A) or bad use of uncertainty boxes

$$\Delta r_e = \frac{r_e^{max} - r_e^{min}}{2} \simeq 1.09\,\text{k}\Omega$$

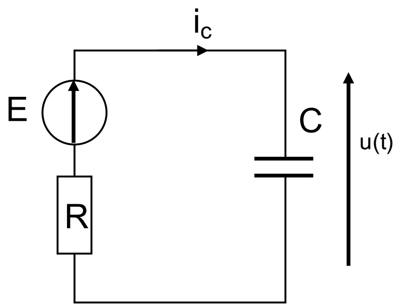
Therefore, keeping only 1 significant digit for the uncertainty we can write:

$$r_e = (10 \pm 1)\,k\Omega$$ (0.25 pt)

2. **0.75 pts**

As K is closed, we can assume that the lamp is still turned OFF. It can be then replaced by an open circuit.

The equivalent circuit is depicted below: (0.25 pt)



initial conditions:

The capacitor being initially discharged and given voltage continuity across the capacitor: $u(0^+) = u(0^-) = u(0) = 0V$ (0.5 pt)

3. **3.25 pts**

Using Kirchhoff's Voltage law:

$$u + Ri_c = E$$ (0.25 pt)

Given $i_c = C\frac{du}{dt}$: (0.25 pt)

$$u + RC\frac{du}{dt} = E$$ (0.5 pt)

The time constant of the circuit is $\tau = RC = 1\,\text{ms}$ so that (0.25+0.25 pt)

$$\frac{du}{dt} + \frac{1}{\tau}u = \frac{E}{\tau} \tag{5}$$

The solution writes:

$$u(t) = u_{gs}(t) + u_{ps}(t)$$

where u_{gs} is a general solution and u_{ps} a particular solution.
u_{gs} is a solution of:

$$\frac{du_{gs}}{dt} + \frac{1}{\tau}u_{gs} = 0$$

hence:

$$u_{gs}(t) = U_0 e^{-t/\tau}, \quad U_0 \in \mathbb{R}$$ (0.5 pt)

A particular solution can be found looking at the steady-state regime ($\frac{du}{dt} = 0$), (5) rewrites:

$$\frac{1}{\tau}u_{ps} = \frac{E}{\tau}$$

hence:

$$u_{ps}(t) = E$$ (0.5 pt)

Finally:

$$u(t > 0) = U_0 e^{-t/\tau} + \frac{E}{R}, \quad U_0 \in \mathbb{R}$$

U_0 can be found using the initial conditions:

$$u(0^+) = u(0^-) = u(0) = 0 \Rightarrow U_0 = -E$$ (0.5 pt)

so that:

$$u(t \geq 0) = E\left(1 - e^{-t/\tau}\right)$$ (0.25 pt)

4. **1.0 pt**

As K is closed $u(t)$ will increase toward $E = 100\,\mathrm{V}$. For a certain time, $u(t)$ will exceed $U_{ON} = 50\,\mathrm{V}$ so that the lamp will turn ON. Since the circuit's configuration will change (the capacitor will be shunted with a parallel resistor), the steady-state cannot be achieved. (0.5 pt)

The time at which the lamp will be turned ON can be found by stating: $u(t_{ON}) = U_{ON}$:

$$E\left(1 - e^{-t_{ON}/\tau}\right) = U_{ON}$$

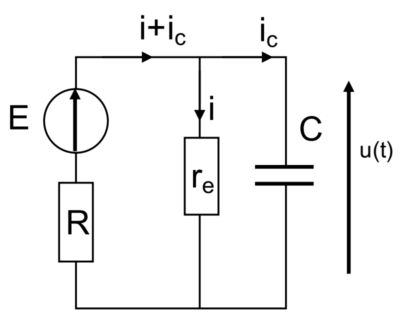
hence:

$$t_{ON} = -\tau \ln\left(1 - \frac{U_{ON}}{E}\right)$$ (0.25 pt)

we find: $t_{ON} \simeq 690\,\mathrm{\mu s}$ (0.25 pt)

5. As the lamp is turned ON, the equivalent circuit is : **1.5 pts**

(0.5 pt)



i being the current flowing in the lamp.

Using Ohm's law and Kirchhoff's voltage laws then have:

$$u(t) = r_e\, i \qquad (0.25\text{ pt})$$

and

$$u(t) + R(i + i_c) = E \qquad (0.25\text{ pt})$$

so that:

$$\left(1 + \frac{R}{r_e}\right) u(t) + Ri_c = E$$

Given $\tau = RC$ and $i_c = C\frac{du}{dt}$:

$$\frac{du}{dt} + \frac{1 + \frac{R}{r_e}}{\tau} u = \frac{E}{\tau} \qquad (6)$$

or any equiv. (0.25 pt)

Equation (6) is valid as the lamp turns ON ($t = 0$). The initial condition is therefore:

$$u(0) = U_{ON} \qquad (0.25\text{ pt})$$

BONUS: **+1.5 pts**

Equation (6) can be rewritten introducing $\tau' = \frac{\tau}{1+\frac{R}{r_e}}$:

$$\frac{du}{dt} + \frac{1}{\tau'} u = \frac{E}{\tau} \qquad (7)$$

Numerical value: $\tau' \simeq r_e C \simeq 10\,\mu\text{s}$

Using the same methodology than in Q3, the solution writes:

$$u(t) = \frac{\tau'}{\tau} E + U_0 e^{-t/\tau'} = \frac{r_e}{R} E + U_0 e^{-t/\tau'}$$

using initial conditions ($u(0) = U_{ON} = \frac{r_e}{R}E + U_0$:

$$u(t) = \frac{r_e}{R} E + \left(U_{ON} - \frac{r_e}{R} E\right) e^{-t/\tau'}$$

$u(t)$ will then decrease toward its steady-state value $u(+\infty) = \frac{r_e}{R}E = 1.0\,\text{V}$. However, before reaching the steady-state regime, $u(t)$ will decrease below $U_{OFF} = 10\,\text{V}$ and the lamp will again be turned OFF. The capacitor will therefore charge again according to Eq. (5) until $u(t)$ will exceed again U_{ON}. The lamp will then be turned ON and $u(t)$ will again be described by Eq.(7)...