TEST 1 - ELEMENTS OF CORRECTION

Exercise 1: (/12)

The package is submitted to its weight only so that Newton's second law leads to:

$$\begin{cases} a_x = 0 \\ a_y = -g \end{cases}$$ 1

The initial speed components are:

$$\begin{cases} v_{x0} = v_0 \sin\alpha \\ v_{y0} = -v_0 \cos\alpha \end{cases}$$ 1

By integration, one obtains:

1 $$\begin{cases} v_x = v_0 \sin\alpha \\ v_y = -gt - v_0 \cos\alpha \end{cases} \quad \text{and} \quad \begin{cases} x = v_0 t \sin\alpha \\ y = -g\dfrac{t^2}{2} - v_0 t \cos\alpha + h \end{cases}$$ 1

1 - The package hits the ground when $y = 0$, hence $0 = -g\dfrac{t^{*2}}{2} - v_0 t^* \cos\alpha + h$ 1

Leading to:

$$t^* = \frac{v_0}{g}\cos\alpha\left(\sqrt{1+\frac{2gh}{v_0^2\cos^2\alpha}}-1\right)$$ (*the negative solution being discarded*) 2

2 – The x-coordinate when the package hits the ground is therefore:

$$x^* = v_0 t^* \sin\alpha = \frac{v_0^2}{2g}\sin 2\alpha\left(\sqrt{1+\frac{2gh}{v_0^2\cos^2\alpha}}-1\right)$$ 2

Noticing that $OA = h\tan\alpha$, the distance to the target at A is:

$$d = OA - x^*$$ 1

3 – Numerical application:

$$OA = 692.8\,m$$
$$x^* = 593.2\,m$$ 2
$$d = 99.6\,m$$

Exercise 2: **(/8)**

Isolate the ball and use Newton's 2nd law:

List of external forces and/or Free-body diagram **1**

$$m\,a\,\mathbf{x} = -mg\,\mathbf{y} + T_A\,\mathbf{u_A} + T_B\,\mathbf{u_B}$$ **2**

($\mathbf{u_A}$ and $\mathbf{u_B}$ are upwards unit vectors in the string directions)

Projecting in the $\mathbf{x}$ - and $\mathbf{y}$ -directions gives:

$$\begin{aligned} m\,a &= T_A\cos(60°) - T_B\cos(60°) \\ 0 &= -mg + T_A\sin(60°) + T_B\sin(60°) \end{aligned}$$ **2**

Hence

$$\begin{aligned} T_A - T_B &= 2m\,a \\ T_A + T_B &= \frac{2}{\sqrt{3}}mg \end{aligned}$$ **1**

Finally giving the two tensions as:

$$\begin{aligned} T_A &= m\left(\frac{g}{\sqrt{3}} + a\right) \\ T_B &= m\left(\frac{g}{\sqrt{3}} - a\right) \end{aligned}$$ **2**

Exercise 3: **(/4)**

The general formulae for velocity and acceleration in polar coordinates are:

$$\begin{aligned} \mathbf{V} &= \dot{r}\,\mathbf{e_r} + r\dot{\theta}\,\mathbf{e_\theta} \\ \mathbf{A} &= \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e_r} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\mathbf{e_\theta} \end{aligned}$$ **1**

1 - Considering the velocity, the only unknown is the angular speed $\dot{\theta}$

The trajectory being vertical, there is no velocity in the horizontal direction (velocity tangent to the trajectory) so that:

$$\mathbf{V}.\mathbf{x} = 0$$
$$\dot{r}\,\mathbf{e_r}.\mathbf{x} + r\dot{\theta}\,\mathbf{e_\theta}.\mathbf{x} = 0 \qquad \textbf{1}$$
$$\dot{r}\cos\theta - r\dot{\theta}\sin\theta = 0$$

From which

$$\dot{\theta} = \frac{\dot{r}}{r}\frac{1}{\tan\theta} \qquad \textbf{1}$$

Numerical application:

$$\dot{\theta} = 0.0834\ rad/s$$

$$V = \|\mathbf{V}\| = \sqrt{350^2 + (5000\times 0.0834)^2} = 544.4\, m/s \qquad \textbf{1}$$

Bonus question– The same technique can be used for acceleration (rectilinear trajectory so no centripetal acceleration) leading to:

$$\mathbf{A}.\mathbf{x} = 0$$
$$\left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e_r}.\mathbf{x} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\mathbf{e_\theta}.\mathbf{x} = 0$$
$$\left(\ddot{r} - r\dot{\theta}^2\right)\cos\theta - \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\sin\theta = 0$$

From which

$$\ddot{\theta} = \frac{\left(\ddot{r} - r\dot{\theta}^2\right)}{r}\frac{1}{\tan\theta} - 2\frac{\dot{r}}{r}\dot{\theta} \qquad \textbf{bonus}$$

Numerical application:

$$\ddot{\theta} = 3.87 10^{-3}\ rad/s^2$$
$$A = \|\mathbf{A}\| = 101.47\, m/s^2$$