Physics exam 3 – Semester 1
February 23rd, 2023. Duration: 1 h

Transient regime of an RC circuit - correction

Part 1 : Theoretical study (~ 9Pts, ~30 min)

1. The typical internal resistance of an LFG is 50 Ω hence R_g can be neglected in front of R. 1.0 Pt

2. Using Kirchhoff voltage law and the constitutive law of capacitors we obtain: 2.0 Pt

$$u_c(t) + \tau \frac{du_c(t)}{dt} = E$$

   with $\tau = (R + R_g)C$ or $\tau = RC$ neglecting R_g in front of R.

   We obtain τ =100 ms

3. Solving the differential equation (homonogeneous +particular solution) **and** finding the integration constant from the continuity of u_c at intial time we obtain: 2.0 Pt

$$u_c(t) = E\left(1 - e^{-t/\tau}\right)$$

   with $i(t) = C\frac{du_c}{dt}$:

$$i(t) = \frac{E}{R} e^{-t/\tau}$$

4. Power received by the resistor:$P_R(t) = R\,i^2(t) = \frac{E^2}{R} e^{-2t/\tau}$ 1.0 Pt
   Energy received (i.e. dissipated) by the resistor:$E_R = \int_0^{t_1} P_R(t)\,dt = \frac{E^2}{R}\int_0^{t_1} e^{-2t/\tau}\,dt$
   We obtain: $E_R = \frac{1}{2}CE^2\left(1 - e^{-2t_1/\tau}\right)$

5. Power received by the capacitor:$P_C(t) = u_c(t)\cdot i(t) = \frac{E^2}{R}\left(e^{-t/\tau} - e^{-2t/\tau}\right)$ 1.0 Pt
   Energy received by the capacitor:$E_C = \int_0^{t_1} P_C(t)\,dt = \frac{E^2}{R}\int_0^{t_1}\left(e^{-t/\tau} - e^{-2t/\tau}\right)dt$
   We obtain: $E_C = \frac{1}{2}CE^2\left(1 + e^{-2t_1/\tau} - 2e^{-t_1/\tau}\right)$

6. Power delivered by the e.m.f. :$P_E(t) = E\cdot i(t) = \frac{E^2}{R} e^{-t/\tau}$ 2.0 Pt
   Energy rdelivered by the e.m.f. :$E_E = \int_0^{t_1} P_E(t)\,dt = \frac{E^2}{R}\int_0^{t_1} e^{-t/\tau}\,dt$
   We obtain: $E_E = CE^2\left(1 - e^{-t_1/\tau}\right)$

   From the expression we obtain for E_E, E_R and E_C we can see that:

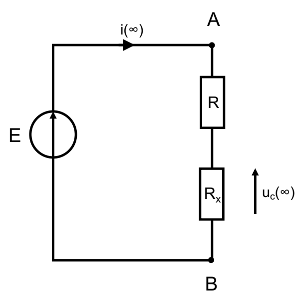
$$E_E = E_R + E_C$$

   which confirms the energy balance.

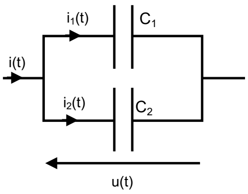
Part 2: experimental study (~ 11Pts, ~30 min)

1. In DC steady-state capacitors are equivalent to open switches. Neglecting R_g the circuit is equivalent to a voltage divider where: 2.0 Pt

$$u_c(t = +\infty) = \frac{R_x}{R + R_x} E$$



2. We consider 2 capacitors (C_1 and C_2) in series, $u(t)$ being the voltage difference across their terminals and i_1/i_2 the current crossing them (PSC convention). Using Kirchhoff current law:



2.0 Pt

$$i = i_1 + i_2$$

with $i_{1,2} = C_{1,2}\frac{du}{dt}$ we obtain:

$$i = (C_1 + C_2)\frac{du}{dt} \Leftrightarrow i = C_{eq}\frac{du}{dt} \quad \text{with } C_{eq} = C_1 + C_2$$

3. Method 1: From Kirchhoff's circuit laws

From the equivalent scheme in transient regime given here we introduce i_c the current running into the equivalent capacitor $(C + C_x)$ and i_{R_x} the one crossing resistor R_x.

From Kirchhoff voltage law: $E = u_c(t) + R\,i(t)$ (1)

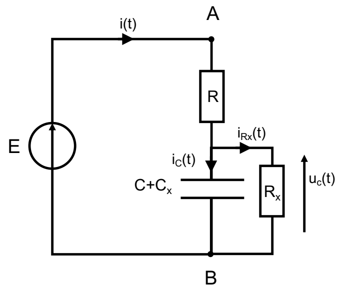
From Kirchhoff current law: $i(t) = i_c(t) + i_{R_x}(t)$ (2)

with $i_c(t) = (C + C_x)\frac{du_c}{dt}$ and $i_{R_x}(t) = \frac{u_c(t)}{R_x}$ we obtain from (2):

$i(t) = (C + C_x)\frac{du_c}{dt} + \frac{u_c(t)}{R_x}$ (3)

Substituting (3) in (1) we obtain:

$\left(1 + \frac{R}{R_x}\right) u_c(t) + R(C + C_x)\frac{du_c}{dt} = E$



3.0 Pt

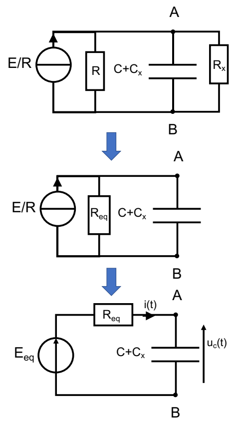
$$\Leftrightarrow u_c(t) + \frac{R(C + C_x)}{1 + \frac{R}{R_x}}\frac{du_c}{dt} = \frac{E}{1 + \frac{R}{R_x}}$$

We finally have:

$$u_c(t) + \tau'\frac{du_c(t)}{dt} = \frac{E}{\alpha} \;(4) \quad \text{with } \alpha = \left(1 + \frac{R}{R_x}\right) \quad \text{and } \tau' = \frac{R}{1 + \frac{R}{R_x}}(C + C_x)$$

Method 2: Using Circuit transformation

We may transform the circuit to an equilavent RC circuit using the successive transformation depicted on the right. We obtain $R_{eq} = \frac{R R_x}{R + R_x}$ and $E_{eq} = \frac{R_x}{R + R_x}E$. The differential equation governing u_c is then:



3.0 Pt

$$u_c(t) + R_{eq}(C + C_x)\,u_c(t) = E_{eq}$$

Substituting E_q and R_eq by their corresponding expressions we retrieve the exact same result.

4. We obtain a very similar equation than in Part1-Q2 whose solution is:

$$u_c(t) = \frac{E}{\alpha}\left(1 - e^{-t/\tau'}\right) \text{(5)}$$

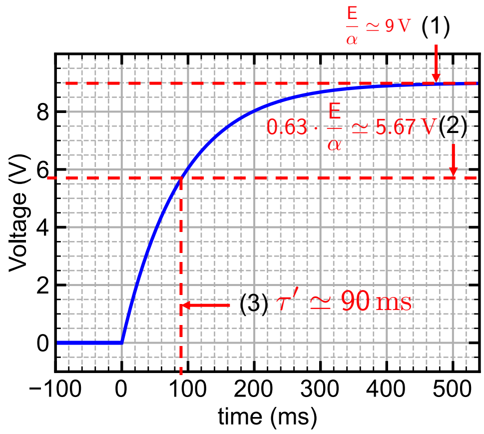
From equation (4) or (5) we can deduce that u_c tends to $\frac{E}{\alpha}$ in steady-state.
Bonus: $\frac{E}{\alpha}$ corresponds to the steady-state value identified in Q1.

From the graph we extract $\boxed{\frac{E}{\alpha} \simeq 9\,\text{V}}$.

τ' can be then extracted knowing u_c is worth $\sim 0.63\frac{E}{\alpha} \simeq 5.7\,\text{V}$ as $t = \tau'$. We obtain $\boxed{\tau' = 90\,\text{ms}}$.
C_x can be neglected in front of C meaning that $\tau' = \frac{R}{1+\frac{R}{R_x}}C$. Knowing τ', C and R we can deduce R_x:

$$\boxed{R_x = \frac{R}{\frac{RC}{\tau'} - 1} = 900\,\text{k}\Omega}$$

R and R_x bring $\alpha \simeq 1.11$
Finally, with the extracted value of $\frac{E}{\alpha}$ we obtain $\boxed{E \simeq 10.0\,\text{V}}$



4.0 Pt
+1.0 Bonus