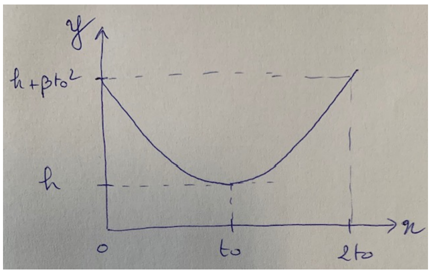
Exercise 1: Kinematics (8 points)

1. for $0 \leq t \leq 2t_0 : \begin{cases} x(t) = \alpha\, t \\ y(t) = h + \beta(t-t_0)^2 \end{cases}$ (1) Given $x(t)$ and $y(t)$ have the dimension of a length, we straightforwardly deduce the dimensions of the parameters: $[h] = L \quad [\alpha] = L.T^{-1} \quad [\beta] = L.T^{-2}$	1.0
2. Substituting t with x/α in the expression of y we obtain: $y = h + \beta\left(\frac{x}{\alpha} - t_0\right)^2$ (1) can be used to determine the coordinates of 3 points of interest (at $t = 0, t_0, 2t_0$): $(x(0), y(0)) = (0, h + \beta\, t_0^2)$ $(x(t_0), y(t_0)) = (\alpha\, t_0, h)$ $(x(2t_0), y(2t_0)) = (2\alpha\, t_0, h + \beta\, t_0^2)$ 	1.0 (y vs x) 1.0 (plot + points of interest)
3. In the cartesian frame: $\vec{v} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} \alpha \\ 2\beta\,(t-t_0) \end{pmatrix}$ (2) $\vec{a} = \begin{pmatrix} \frac{d^2x}{dt^2} \\ \frac{d^2y}{dt^2} \end{pmatrix} = \begin{pmatrix} 0 \\ 2\beta \end{pmatrix}$ (3)	1.0
4. In the local Frame $(\vec{T}, \vec{N})$: $\vec{v} = \|\|\vec{v}\|\|\,\vec{T} \quad \text{with} \quad \|\|\vec{v}\|\| = \sqrt{\alpha^2 + 4\beta^2(t-t_0)^2}$ we obtain: $\vec{v} = \sqrt{\alpha^2 + 4\beta^2(t-t_0)^2} \cdot \vec{T}$ $\vec{a} = \frac{d\|\|\vec{v}\|\|}{dt}\vec{T} + \frac{\|\|\vec{v}\|\|^2}{R_c}\vec{N}$ R_c being the local radius of curvature (as introduced in Q5).	2.0

we obtain: $\vec{a} = \frac{4\beta^2(t-t_0)}{\sqrt{\alpha^2+4\beta^2(t-t_0)^2}} \cdot \vec{T} + \frac{\alpha^2+4\beta^2(t-t_0)^2}{R_c} \cdot \vec{N}$ in summary, writing $\vec{v}$ and $\vec{a}$ as row vectors in the local frame: $\vec{v}_{\text{loc}} = \begin{pmatrix} \sqrt{\alpha^2+4\beta^2(t-t_0)^2} \\ 0 \end{pmatrix} \quad \vec{a}_{\text{loc}} = \begin{pmatrix} \frac{4\beta^2(t-t_0)}{\sqrt{\alpha^2+4\beta^2(t-t_0)^2}} \\ \frac{\alpha^2+4\beta^2(t-t_0)^2}{R_c} \end{pmatrix}$ (4)	
2nd version, without introducing R_c From the plot of the trajectory, we see that $\vec{N} = \vec{u}_z \times \vec{T}$ (center of curvature above the curve). Noting $v = \sqrt{\alpha^2+4\beta^2(t-t_0)^2}$, we have $\vec{T} = \frac{\alpha}{v}\vec{u}_x + \frac{2\beta(t-t_0)}{v}\vec{u}_y, \quad \vec{N} = -\frac{2\beta(t-t_0)}{v}\vec{u}_x + \frac{\alpha}{v}\vec{u}_y$ Velocity: $\vec{v} = \sqrt{\alpha^2+4\beta^2(t-t_0)^2}\vec{T}$ Acceleration tangential component: $a_T = \frac{d\Vert\vec{v}\Vert}{dt}(\text{or} = 2\beta\vec{u}_y \cdot \vec{T}) = \frac{4\beta^2(t-t_0)}{\sqrt{\alpha^2+4\beta^2(t-t_0)^2}}$ Acceleration normal component $a_N = \vec{a} \cdot \vec{N} = \frac{2\alpha\beta}{\sqrt{\alpha^2+4\beta^2(t-t_0)^2}}$ If no idea for $\vec{N}$, one needs to write $a_N = \Vert\vec{a} - a_T\vec{T}\Vert$, which is (slightly) longer...	**or** 1 0.5 0.5
5. The radius of curvature can be determined by equating the norm of the acceleration expressed both in the cartesian (3) and the local frame (4): We have $\vec{a}_{\text{car}}(t=t_0) = \begin{pmatrix} 0 \\ 2\beta \end{pmatrix} \quad \text{and} \quad \vec{a}_{\text{loc}}(t=t_0) = \begin{pmatrix} 0 \\ \frac{\alpha^2}{R_c} \end{pmatrix}$ (5) therefore: $\Vert\vec{a}_{\text{car}}(t=t_0)\Vert = \Vert\vec{a}_{\text{loc}}(t=t_0)\Vert \iff 2\beta = \frac{\alpha^2}{Rc}$ we obtain: $R_c = \frac{1}{2}\frac{\alpha^2}{\beta}$	1.0 (expressions of $\vec{a}(t_0)$) 1.0 (Rc)

Exercise 2: Flow force measure (12 points)

1. With a flow (drag) force, the rotation is prevented by the force of the weighing scale on the system. This force is measured through the mass displayed, which can allow to infer F_D	0.5
2. Newton's third law: $\vec{F}_{\text{scale}\to\text{system}} = -\vec{F}_{\text{system}\to\text{scale}}$ mass displayed $m = \frac{\Vert\vec{F}_{\text{system}\to\text{scale}}\Vert}{g}$ $\Leftrightarrow \Vert\vec{F}_{\text{scale}\to\text{system}}\Vert = mg$	0.5 0.5
3. Configuration of figure (a). System {Bars + obstacle}, Terrestrial reference frame (Galilean) External forces: * Weight of bar OA+ obstacle: $\vec{W}_1 = -M_1 g\vec{e}_y$, on G_1 * Weight of bar OB: $\vec{W}_2 = -M_2 g\vec{e}_y$, on G_2 * Force of the scale: $\vec{F}_B = +m_0 g\vec{e}_y$, on B * Reaction force on the pivot axis: $\vec{R}_O$, on O System at equilibrium: $\Sigma\vec{F}_{\text{ext}} = \vec{0}$ and $\Sigma\mathcal{M}_{\text{ext}}(Oz) = 0$ $\mathcal{M}_{\vec{W}_1}(Oz) = 0$ (vertical axis) $\mathcal{M}_{\vec{W}_2}(Oz) = +\frac{L_2}{2} M_2\ g$ $\mathcal{M}_{\vec{F}_B}(Oz) = -L_2\ m_0\ g$ $\mathcal{M}_{\vec{R}_O}(Oz) = 0$ Equilibrium of the moments gives $m_0 = M_2/2$	0.5 1.5 (-0.5 per missing force) 0.5 1(accept vectorial form) 0.5
4. Same system, with one additional force: fluid force $\vec{F}_D = F_D\vec{e}_x$ on C The system is now tilted, the moments are $\mathcal{M}_{\vec{W}_1}(Oz) = -\frac{2L_1}{3} M_1\ g\sin\theta$ $\mathcal{M}_{\vec{W}_2}(Oz) = +\frac{L_2}{2} M_2\ g\cos\theta = L_2 m_0\ g\cos\theta$ $\mathcal{M}_{\vec{F}_B}(Oz) = -L_2\ m\ g\cos\theta$ $\mathcal{M}_{\vec{F}_D}(Oz) = +OC\ F_D\cos\theta = +(L_1 - h/2)F_D\cos\theta$ $\mathcal{M}_{\vec{R}_O}(Oz) = 0$ Equilibrium: $-\frac{2L_1}{3} M_1\ g\sin\theta + \frac{L_2}{2} M_2\ g\cos\theta - L_2\ m_0\ g\cos\theta + (L_1 - \frac{h}{2})F_D\cos\theta = 0$ $\Leftrightarrow F_D = \frac{L_2\ g(m-m_0)}{L_1 - h/2} + \frac{2}{3}\ \frac{L_1 M_1\ g}{L_1 - h/2}\tan\theta$ (so that $a = L_2 g$ and $b = \frac{2}{3}\ \frac{L_1 M_1\ g}{L_1 - h/2}$)	0.5 1.5 0.5 0.5
5. θ is related to the vertical position of B Noting $\Delta y_B = 1$ mm, we have $\sin\theta_{\max} = \frac{\Delta y_B}{L_2}$ $\Rightarrow \theta_{\max} = 0.005 = 0.29°$ Relative error: calculation from expression of Q4 (accept $\varepsilon = \frac{OC}{m-m_0}\ \frac{b}{a}\tan\theta_{\max}$ as literal expression) Numerical application for $m - m_0 = 400g$: $\varepsilon = 3.0\%$, negligible. *Bonus for a remark that even for an extremely small angle the relative error is not that small – this is due to the heavy vertical system*	 0.5 0.5 0.5 *0.5*
6. In this case, $F_D = \frac{a}{OC}(m - m_0)$ OC is known with an uncertainty $\Delta OC = \frac{h}{2}$ Differential method: $\frac{\Delta F_D}{F_D} = \frac{\Delta OC}{OC}$	 0.5 0.5
7. $OC = L_1 - h/2$ We want $\frac{\Delta F_D}{F_D} = \varepsilon_{\max} = 5\% = \frac{\Delta OC}{OC}$ $\Rightarrow OC = \frac{\Delta OC}{\varepsilon_{\max}} \Leftrightarrow L_1 = \frac{h}{2}\left(1 + \frac{1}{\varepsilon_{\max}}\right)$ Numerical application: $L_1 = 1.75$ m *Bonus for a comment that this is the minimum length required, and also if any relevant comment regarding this large value!*	 0.5 0.5