SCAN 2 — Solution of Math Test #3

March 31, 2016

Romaric Pujol, romaric.pujol@insa-lyon.fr

Exercise 1.

1. Let $x \in \mathbb{C}^*$ and define
$$\forall n \in \mathbb{N},\ u_n = \cosh(n)|x|^n > 0.$$
We use the ratio test for the convergence of the series $\sum_n u_n$:
$$\frac{u_{n+1}}{u_n} = \frac{\cosh(n+1)}{\cosh(n)}|x| = \frac{\mathrm{e}^{n+1} - \mathrm{e}^{-n-1}}{\mathrm{e}^n - \mathrm{e}^{-n}}|x| \underset{n\to+\infty}{\sim} \frac{\mathrm{e}^{n+1}}{\mathrm{e}^n}|x| = \mathrm{e}|x| \underset{n\to+\infty}{\longrightarrow} \mathrm{e}|x|.$$
Hence, by the ratio test, the series $\sum_n u_n$ converges if $|x| < 1/\mathrm{e}$ and diverges if $|x| > 1/\mathrm{e}$: we conclude that $R = 1/\mathrm{e}$.
Now, for $x \in (-R, R)$,
$$\begin{aligned}\sum_{n=0}^{+\infty} \cosh(n)x^n &= \sum_{n=0}^{+\infty} \frac{\mathrm{e}^n + \mathrm{e}^{-n}}{2}x^n = \frac{1}{2}\sum_{n=0}^{+\infty}\left((\mathrm{e}x)^n + \left(\frac{x}{\mathrm{e}}\right)^n\right)\\ &= \frac{1}{2}\left(\sum_{n=0}^{+\infty}(\mathrm{e}x)^n + \sum_{n=0}^{+\infty}\left(\frac{x}{\mathrm{e}}\right)^n\right) \qquad \textit{since both series converge}\\ &= \frac{1}{2}\left(\frac{1}{1-\mathrm{e}x} + \frac{1}{1-x/\mathrm{e}}\right)\end{aligned}$$

2. Let $x \in \mathbb{C}^*$ and define
$$\forall n \in \mathbb{N},\ u_n = \frac{n}{(2n+1)!}|x|^{2n} > 0.$$
We use the ratio test for the convergence of the series $\sum_n u_n$:
$$\frac{u_{n+1}}{u_n} = \frac{n+1}{(2n+3)!}\frac{(2n+1)!}{n}|x^2| = \frac{n+1}{(2n+2)(2n+3)n}|x^2| \underset{n\to+\infty}{\longrightarrow} 0.$$
Hence, by the ratio test, the series $\sum_n u_n$ converges. We conclude that $R = +\infty$.
For $x \in \mathbb{R}^*$,
$$\begin{aligned}\sum_{n=0}^{+\infty}(-1)^n\frac{n}{(2n+1)!}x^{2n} &= \frac{1}{2}\sum_{n=0}^{+\infty}(-1)^n\frac{(2n+1)-1}{(2n+1)!}x^{2n} = \frac{1}{2}\sum_{n=0}^{+\infty}(-1)^n\left(\frac{1}{(2n)!} - \frac{1}{(2n+1)!}\right)x^{2n}\\ &= \frac{1}{2}\left(\sum_{n=0}^{+\infty}(-1)^n\frac{1}{(2n)!}x^{2n} - \frac{1}{x}\sum_{n=0}^{+\infty}(-1)^n\frac{1}{(2n+1)!}x^{2n+1}\right) \qquad \textit{since both series converge}\\ &= \frac{1}{2}\cos(x) - \frac{1}{2x}\sin(x).\end{aligned}$$
If $x = 0$, we obtain:
$$\sum_{n=0}^{+\infty}(-1)^n\frac{n}{(2n+1)!}0^{2n} = (-1)^0\frac{0}{1!} = 0.$$

Exercise 2. First observe that the function
$$\begin{aligned}(1,+\infty) &\longrightarrow \mathbb{R}\\ x &\longmapsto \frac{1}{x\ln(x)}\end{aligned}$$
is decreasing (and piecewise continuous). Hence, by the integral comparison test, for $N \geq 3$,
$$\int_2^{N+1}\frac{\mathrm{d}x}{x\ln(x)} \leq \sum_{n=2}^{N}\frac{1}{n\ln(n)} = \frac{1}{2\ln(2)} + \sum_{n=3}^{N}\frac{1}{n\ln(n)} \leq \frac{1}{2\ln(2)} + \int_2^N\frac{\mathrm{d}x}{x\ln(x)}.$$

Since

$$\int \frac{\mathrm{d}x}{x\ln(x)} = \ln(\ln x) + C,$$

we obtain:

$$\forall N \geq 3,\ \ln\bigl(\ln(N+1)\bigr) - \ln(\ln 2) \leq S_N \leq \frac{1}{2\ln(2)} + \ln(\ln N) - \ln(\ln 2).$$

In particular, by the Squeeze Theorem, we conclude that

$$\lim_{N\to+\infty} S_N = +\infty,$$

hence the series (S) diverges. Now, for $N \geq 3$,

$$\frac{\ln\bigl(\ln(N+1)\bigr) - \ln(\ln 2)}{\ln(\ln N)} \leq \frac{S_N}{\ln(\ln N)} \leq 1 + \frac{\dfrac{1}{2\ln(2)} - \ln(\ln 2)}{\ln(\ln N)},$$

and since $\ln\bigl(\ln(N+1)\bigr) \underset{N\to+\infty}{\sim} \ln(\ln N)$ (see below) we conclude, by the Squeeze Theorem, that

$$\lim_{N\to+\infty} \frac{S_N}{\ln(\ln N)} = 1,$$

i.e.,

$$S_N \underset{N\to+\infty}{\sim} \ln(\ln N).$$

To show that $\ln\bigl(\ln(N+1)\bigr) \underset{N\to+\infty}{\sim} \ln(\ln N)$, we may use the Mean Value Theorem: for $N \geq 3$, there exists $c_N \in (N, N+1)$ such that

$$\ln\bigl(\ln(N+1)\bigr) = \ln(\ln N) + \frac{1}{c_N \ln c_N},$$

hence

$$\frac{\ln\bigl(\ln(N+1)\bigr)}{\ln(\ln N)} = 1 + \frac{1}{c_N \ln c_N \ln(\ln N)} \underset{N\to+\infty}{\longrightarrow} 1.$$

Exercise 3.

1. Notice that

$$\begin{aligned}
u_{n+1} - u_n &= -\ln(n+1) + \sum_{k=1}^{n+1} \frac{1}{k} + \ln(n) - \sum_{k=1}^{n} \frac{1}{k} \\
&= \ln\left(\frac{n}{n+1}\right) + \frac{1}{n+1} \\
&= \ln\left(1 - \frac{1}{n+1}\right) + \frac{1}{n+1} \\
&\underset{n\to+\infty}{=} -\frac{1}{n+1} - \frac{1}{2(n+1)^2} + o\left(\frac{1}{n^2}\right) \ + \ \frac{1}{n+1} \\
&\underset{n\to+\infty}{\sim} -\frac{1}{2n^2} < 0,
\end{aligned}$$

   which is the general term of a convergent series. Hence, by the equivalent test, the series $\sum_n (u_{n+1} - u_n)$ is convergent.

2. Observe that for $N \in \mathbb{N}$,

$$\sum_{n=1}^{N} (u_{n+1} - u_n) = u_{N+1} - u_1 = u_{N+1} - 1.$$

   Now, since the series $\sum_n (u_{n+1} - u_n)$ converges, i.e., the limit

$$\lim_{N\to+\infty} \sum_{n=1}^{N} (u_{n+1} - u_n) = \lim_{N\to+\infty} u_{N+1} - 1$$

   exists in $\mathbb{R}$, we conclude that $\lim_{N\to+\infty} u_{N+1}$ exists in $\mathbb{R}$, and hence that the sequence $(u_n)n \geq 1$ converges.

3. For $N \in \mathbb{N}$ with $N \geq 2$:

$$S_N = \sum_{n=2}^{N} \frac{1}{n} - \ln\left(\frac{n}{n-1}\right) = \sum_{n=2}^{N} \frac{1}{n} - \big(\ln(n) - \ln(n-1)\big) = \left(\sum_{n=2}^{N} \frac{1}{n}\right) - \ln(N) = u_N - 1 \underset{N\to+\infty}{\longrightarrow} \gamma - 1$$

hence we conclude that the series (S) converges and that

$$\sum_{n=2}^{+\infty} \left(\frac{1}{n} - \ln\left(\frac{n}{n-1}\right)\right) = \gamma - 1.$$

Exercise 4. Define the sequence $(u_n)_{n\in\mathbb{N}}$ as

$$u_n = \frac{(-1)^n}{1+\sqrt{n}}.$$

Clearly, the sequence $(u_n)_{n\in\mathbb{N}}$ is an alternating sequence, and for $n \in \mathbb{N}$ one has

$$|u_{n+1}| < |u_n|.$$

Moreover,

$$\lim_{n\to+\infty} u_n = 0,$$

hence, by the Alternating Series Test, the series $\sum_n u_n$ is convergent. We also know that

$$\forall N \geq 0, |R_N| = \left|\sum_{n=N+1}^{+\infty} u_n\right| \leq |u_{N+1}| = \frac{1}{1+\sqrt{N+1}}.$$

Hence, a sufficient condition for S_N to be an approximation of the sum of the series (S) with error less than 10^{-5} is

$$\frac{1}{1+\sqrt{N+1}} < 10^{-5} \iff 1+\sqrt{N+1} > 10^5 \iff N > (10^5-1)^2 - 1.$$

Hence, a sufficient condition is $N \geq 10^{10}$.

Exercise 5.

1. Let $a, b \in \mathbb{R}_+^*$. The function

$$t \longmapsto \frac{\mathrm{e}^{-at} - \mathrm{e}^{-bt}}{t}$$

is continuous on $\mathbb{R}_+^*$, hence the integral $I(a,b)$ is improper at 0^+ and at $+\infty$. Now,

$$\frac{\mathrm{e}^{-at} - \mathrm{e}^{-bt}}{t} \underset{t\to0}{=} \frac{1}{t}\Big(\big(1 - at + o(t)\big) - \big(1 - bt + o(t)\big)\Big) \underset{t\to0}{=} -a + b + o(1) \underset{t\to0}{\longrightarrow} -a + b \in \mathbb{R},$$

hence the integral $I(a,b)$ is falsely improper at 0^+. Moreover,

$$\forall t \in [1,+\infty), \quad \left|\frac{\mathrm{e}^{-at} - \mathrm{e}^{-bt}}{t}\right| \leq \mathrm{e}^{-at} + \mathrm{e}^{-bt},$$

and since $a > 0$ and $b > 0$, we know that the improper integrals

$$\int_1^{+\infty} \mathrm{e}^{-at}\,\mathrm{d}t \qquad \text{and} \qquad \int_1^{+\infty} \mathrm{e}^{-bt}\,\mathrm{d}t$$

converge hence, by the comparison test, the improper integral $I(a,b)$ is absolutely convergent at $+\infty$. Hence the improper integral $I(a,b)$ converges.

2. Let $A > 0$. We use Theorem 3 with $I = \mathbb{R}_+^*$, $J = [A,+\infty)$ and the function u defined by:

$$u : \begin{array}{ccc} \mathbb{R}_+^* \times [A,+\infty) & \longrightarrow & \mathbb{R} \\ (t,x) & \longmapsto & \dfrac{\mathrm{e}^{-t} - \mathrm{e}^{-xt}}{t}. \end{array}$$

   - For all $(t,x) \in \mathbb{R}_+^* \times [A,+\infty)$ the partial derivative $\partial_2 u(t,x)$ clearly exists and in fact:

$$\partial_2 u(t,x) = \mathrm{e}^{-xt}.$$

- Clearly, for all $x \in [A, +\infty)$, the functions

$$\begin{array}{rcl} \mathbb{R}_+^* & \longrightarrow & \mathbb{R} \\ t & \longmapsto & u(t,x) = \dfrac{\mathrm{e}^{-t} - \mathrm{e}^{-xt}}{t} \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \mathbb{R}_+^* & \longrightarrow & \mathbb{R} \\ t & \longmapsto & \partial_2 u(t,x) = \mathrm{e}^{-xt} \end{array}$$

  are continuous.
- Define the function g as

$$g\ :\ \begin{array}{rcl} \mathbb{R}_+^* & \longrightarrow & \mathbb{R} \\ t & \longmapsto & \mathrm{e}^{-At}. \end{array}$$

  Clearly, g is continuous and

$$\forall (t,x) \in \mathbb{R}_+^* \times [A, +\infty),\ \left|\partial_2 u(t,x)\right| = \mathrm{e}^{-xt} \le \mathrm{e}^{-At}.$$

- Moreover, the improper integral

$$\int_0^{+\infty} \mathrm{e}^{-At}\,\mathrm{d}t$$

  converges (since $A > 0$). We also know (from Question 1) that for all $x \in \mathbb{R}_+^*$, the improper integral

$$\int_0^{+\infty} u(t,x)\,\mathrm{d}t$$

  converges.

Hence we conclude that φ is differentiable on $[A, +\infty)$ and that

$$\forall x \in [A, +\infty),\ \varphi'(x) = \int_0^{+\infty} \mathrm{e}^{-xt}\,\mathrm{d}t.$$

Now, since this is true for all $A > 0$, we conclude that φ is differentiable on

$$\bigcup_{A \in \mathbb{R}_+^*} [A, +\infty) = \mathbb{R}_+^*$$

and that

$$\forall x \in \mathbb{R}_+^*,\ \varphi'(x) = \int_0^{+\infty} \mathrm{e}^{-xt}\,\mathrm{d}t.$$

Let $X > 0$ and $x \in \mathbb{R}_+^*$. Then:

$$\int_0^X \mathrm{e}^{-xt}\,\mathrm{d}t = \left[\frac{\mathrm{e}^{-xt}}{-x}\right]_{t=0}^{t=X} = -\frac{\mathrm{e}^{-xX}}{x} + \frac{1}{x} \underset{X \to +\infty}{\longrightarrow} \frac{1}{x}.$$

Hence

$$\forall x \in \mathbb{R}_+^*,\ \varphi'(x) = \frac{1}{x}.$$

3. From the previous question we conclude that there exists $C \in \mathbb{R}$ such that

$$\forall x \in \mathbb{R}_+^*,\ \varphi(x) = \ln(x) + C.$$

Now, since

$$\varphi(1) = \int_0^{+\infty} \frac{\mathrm{e}^{-t} - \mathrm{e}^{-t}}{t}\,\mathrm{d}t = \int_0^{+\infty} 0\,\mathrm{d}t = 0,$$

we conclude that $C = 0$, hence

$$\varphi\ :\ \begin{array}{rcl} \mathbb{R}_+^* & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \ln(x). \end{array}$$

4. Let $a, b \in \mathbb{R}_+^*$ and $X, Y \in \mathbb{R}_+^*$ such that $X < Y$. Using the substitution $s = at$:

$$\int_X^Y \frac{\mathrm{e}^{-at} - \mathrm{e}^{-bt}}{t}\,\mathrm{d}t = \int_{aX}^{aY} \frac{\mathrm{e}^{-s} - \mathrm{e}^{-bs/a}}{s/a}\,\frac{\mathrm{d}s}{a} \underset{\substack{X \to 0^+ \\ Y \to +\infty}}{\longrightarrow} \int_0^{+\infty} \frac{\mathrm{e}^{-s} - \mathrm{e}^{-bs/a}}{s}\,\mathrm{d}s = \varphi(b/a) = \ln(b/a).$$

Hence

$$\forall a, b \in \mathbb{R}_+^*,\ I(a,b) = \int_0^{+\infty} \frac{\mathrm{e}^{-at} - \mathrm{e}^{-bt}}{t}\,\mathrm{d}t = \ln\left(\frac{b}{a}\right).$$