INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — Math Test #3 — 2h

March 31, 2016

No documents, no calculators, no cell phones or electronic devices allowed but you may keep your pet blobfish for moral support.

All your answers must be fully justified, unless noted otherwise.

For your convenience, we have recalled the main theorems of integral calculus at the end of this document.

Exercise 1. For each of the following power series, determine its radius of convergence R and determine an explicit expression (without the $\sum$ symbol) for $x \in (-R, R)$.

$$(1)\ \sum_{n=0}^{+\infty} \cosh(n) x^n, \qquad (2)\ \sum_{n=0}^{+\infty} (-1)^n \frac{n}{(2n+1)!} x^{2n}.$$

Exercise 2. Use the integral comparison test to determine the nature of the series

$$\text{(S)} \qquad \sum_n \frac{1}{n\ln(n)}$$

and to find an equivalent of the partial sum

$$S_N = \sum_{n=2}^{N} \frac{1}{n\ln(n)}$$

as $N \to +\infty$.

Exercise 3. We define the sequence $(u_n)_{n\geq 1}$ as

$$\forall n \in \mathbb{N}^*,\ u_n = -\ln(n) + \sum_{k=1}^{n} \frac{1}{k}.$$

1. Show that the series $\sum_n (u_{n+1} - u_n)$ converges.
2. Deduce that the the sequence $(u_n)_{n\geq 1}$ converges; we denote by γ the value of its limit.[1]
3. Show that the series

   $$\text{(S)} \qquad \sum_n \left(\frac{1}{n} - \ln\left(\frac{n}{n-1}\right)\right)$$

   converges and determine the value of the sum

   $$\sum_{n=2}^{+\infty} \left(\frac{1}{n} - \ln\left(\frac{n}{n-1}\right)\right)$$

   in terms of γ. Hint: express the partial sum of this series in terms of the sequence $(u_n)_{n\geq 1}$.

Exercise 4. Show that the series

$$\text{(S)} \qquad \sum_n \frac{(-1)^n}{1+\sqrt{n}}$$

converges. Consider the partial sums of the series (S):

$$\forall N \in \mathbb{N},\ S_N = \sum_{n=0}^{N} \frac{(-1)^n}{1+\sqrt{n}}.$$

Find a value of $N \in \mathbb{N}$ for which S_N is an approximation of the sum of the series (S) with error less than 10^{-5}.

[1] the number γ is known as the *Euler–Mascheroni constant.*

Exercise 5.

1. Show that for all $a, b \in \mathbb{R}_+^*$ the improper integral

$$I(a,b) = \int_0^{+\infty} \frac{\mathrm{e}^{-at} - \mathrm{e}^{-bt}}{t} \,\mathrm{d}t$$

   converges.

2. We define the function

$$\begin{aligned} \varphi \;:\; \mathbb{R}_+^* &\longrightarrow \mathbb{R} \\ x &\longmapsto I(1,x) = \int_0^{+\infty} \frac{\mathrm{e}^{-t} - \mathrm{e}^{-xt}}{t} \,\mathrm{d}t. \end{aligned}$$

   Show that φ is differentiable on $\mathbb{R}_+^*$ and, for $x \in \mathbb{R}_+^*$, find an explicit expression of $\varphi'(x)$.

3. Deduce an explicit expression of φ.

4. Let $a, b \in \mathbb{R}_+^*$. Use an appropriate substitution to compute the value of $I(a,b)$.

Theorem 1 (*Dominated Convergence Theorem*). *Let I be a real interval with endpoints a and b in $\overline{\mathbb{R}}$ (with $a < b$). Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of piecewise continuous functions on I such that:*

- *the sequence of functions $(f_n)_{n\in\mathbb{N}}$ converges pointwise on I to a piecewise continuous function f,*
- *there exists a piecewise continuous function g on I such that*
$$\forall t \in I,\ \forall n \in \mathbb{N},\ \big|f_n(t)\big| \leq g(t),$$
and, in the case where I is not a closed and bounded interval, the improper integral
$$\int_a^b g(t)\,\mathrm{d}t$$
converges (the domination condition*).*

Then
$$\lim_{n\to+\infty} \int_a^b \big|f_n(t) - f(t)\big|\,\mathrm{d}t = 0,$$
and in particular,
$$\lim_{n\to+\infty} \int_a^b f_n(t)\,\mathrm{d}t = \int_a^b f(t)\,\mathrm{d}t,$$
and, in the case where I is not a closed and bounded interval, the improper integral on the right-hand side is absolutely convergent.

Theorem 2 (*Limits*). *Let I be a real interval with endpoints a and b in $\overline{\mathbb{R}}$ (with $a < b$) and let J be a real interval, let $x_0 \in \overline{J}$, and let $u : I \times J \longrightarrow \mathbb{R}$ be a function such that for all $x \in J$, the function*
$$\begin{array}{rcl} I & \longrightarrow & \mathbb{R} \\ t & \longmapsto & u(t,x) \end{array}$$
is piecewise continuous and such that there exists a piecewise continuous function $v : J \longrightarrow \mathbb{R}$ such that
$$\forall t \in I,\ \lim_{x\to x_0} u(t,x) = v(t).$$
Assume that there exists a function $g : I \longrightarrow \mathbb{R}$ such that
$$\forall t \in I,\ \forall x \in J,\ \big|u(t,x)\big| \leq g(t),$$
and, in the case where I is not a closed and bounded interval, the improper integral
$$\int_a^b g(t)\,\mathrm{d}t$$
converges (the domination condition*). Then,*
$$\lim_{x\to x_0} \int_a^b u(t,x)\,\mathrm{d}t = \int_a^b v(t)\,\mathrm{d}t.$$

Theorem 3 (*Differentiability*). *Let I be a real interval with endpoints a and b in $\overline{\mathbb{R}}$, let J be real interval, and let $u : I \times J \longrightarrow \mathbb{R}$ be a function such that:*

- *the partial derivative $\partial_2 u$ exists on $I \times J$,*
- *for all $x \in J$, the functions*
$$\begin{array}{rcl} I & \longrightarrow & \mathbb{R} \\ t & \longmapsto & u(t,x) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} I & \longrightarrow & \mathbb{R} \\ t & \longmapsto & \partial_2 u(t,x) \end{array}$$
are piecewise continuous,
- *there exists a piecewise continuous function $g : I \longrightarrow \mathbb{R}$ such that*
$$\forall t \in I,\ \forall x \in J,\ \big|\partial_2 u(t,x)\big| \leq g(t),$$
- *in the case where I is not a closed and bounded interval, the improper integral*
$$\int_a^b g(t)\,\mathrm{d}t$$
converges (the domination condition*) and there exists $x_0 \in J$ such that the improper integral*
$$\int_a^b u(t,x_0)\,\mathrm{d}t$$
converges.[2]

Then, the function f defined as
$$\begin{array}{rrcl} f : & J & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \displaystyle\int_a^b u(t,x)\,\mathrm{d}t \end{array}$$
is differentiable on J and
$$\forall x \in J,\ f'(x) = \int_a^b \partial_2 u(t,x)\,\mathrm{d}t.$$

[2]This condition can be replaced by
$$\forall x \in J,\ \text{the improper integral} \int_a^b u(t,x)\,\mathrm{d}t \text{ converges.}$$