INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — DS 2 — 3h

June 17, 2016

Exercise 1 (*8.5 points*)**.** The two parts of this exercise can be solved independently, using the results of Part I in Part II.

Part I – *Study of a power series expansion — 4 points*
We define the function

$$p\ :\ \begin{array}{ccc}(-1,1) & \longrightarrow & \mathbb{R}\\ u & \longmapsto & \sqrt{1+u}.\end{array}$$

1. Justify that p is a solution on I of the following differential equation

   (E) $$(2+2u)y'(u)-y(u)=0$$

   such that $y(0)=1$.

   We recall that p is the only solution of Equation (E) on $(-1,1)$ that takes the value 1 at 0.

2. By looking for a solution of the differential equation (E) such that $y(0)=1$ in the form of a function y that possesses a power series expansion, say

   $$y(u)=\sum_{n=0}^{+\infty} b_n u^n,$$

   show that:

   - the function p possesses a power series expansion (and specify its radius of convergence);
   - the power series expansion of p is

     $$p(u)=\sum_{n=0}^{+\infty} b_n u^n$$

     where $b_0=1$ and

     $$\forall n\in\mathbb{N},\ b_{n+1}=-\frac{2n-1}{2(n+1)}b_n.$$

3. Show that:

   $$\forall n\in\mathbb{N},\ b_n=(-1)^{n-1}\frac{(2n)!}{(2n-1)2^{2n}(n!)^2}.$$

4. Prove that the sequence $\big(|b_n|\big)_{n\in\mathbb{N}^*}$ is a non-increasing sequence.

5. We admit the following equivalent (known as *Stirling's formula*) :

   $$n!\underset{n\to+\infty}{\sim}\sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n.$$

   Show, using Stirling's formula, that

   $$|b_n|\underset{n\to+\infty}{\sim}\frac{1}{\alpha n^\beta}$$

   where α and β are two real numbers strictly greater than 1.

Part II – *An approximation an arc length— 4.5 points*
Let $a,b\in\mathbb{R}$ such that $a<b$. For a function $f:[a,b]\longrightarrow\mathbb{R}$ of class C^1 we define

$$L(f)=\int_a^b\sqrt{1+f'(t)^2}\,\mathrm{d}t.$$

We recall that this number $L(f)$ is the arc length of the graph of f. The goal of this part is to determine an approximation of the arc length of the hyperbola, graph of the following function:

$$f\ :\ \begin{array}{ccc}[1/2,1] & \longrightarrow & \mathbb{R}\\ t & \longmapsto & 1/t.\end{array}$$

We're hence in the case where $a=1/2$ et $b=1$.

1. Explicit the integral expression of $L(f)$.

2. The goal of this question is to obtain an expression of $L(f)$ in the form of a series.

   a) Show that

$$\forall t \in (0,1), \quad \frac{\sqrt{1+t^4}}{t^2} = \frac{1}{t^2} + \sum_{n=1}^{+\infty} b_n t^{4n-2},$$

   where the sequence $(b_n)_{n\in\mathbb{N}^*}$ is the sequence defined in the previous part.

   b) We define the sequence of function $(S_N)_{N\in\mathbb{N}^*}$ by:

$$\forall N \in \mathbb{N}^*, \quad S_N : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ t & \longmapsto & \displaystyle\sum_{n=1}^{N} b_n t^{4n-2}. \end{array}$$

      i) Show that the series $\big(S_N(1)\big)_{N\in\mathbb{N}^*}$ converges absolutely.

      ii) Using the Dominated Convergence Theorem (that is recalled below) applied to the sequence of functions $(S_N)_{N\in\mathbb{N}^*}$ on $[1/2, 1)$, show that

$$\int_{1/2}^{1} \sum_{n=1}^{+\infty} b_n t^{4n-2}\,\mathrm{d}t = \sum_{n=1}^{+\infty} \int_{1/2}^{1} b_n t^{4n-2}\,\mathrm{d}t.$$

   c) Deduce that

$$L(f) = 1 + \sum_{n=1}^{+\infty} \frac{b_n}{4n-1}\left(1 - \frac{1}{2^{4n-1}}\right).$$

3. Explain why an upper bound of the error committed by approximating $L(f)$ by

$$1 + \sum_{n=1}^{4} \frac{b_n}{4n-1} - \sum_{n=1}^{4} \frac{b_n}{2^{4n-1}(4n-1)}$$

   is

$$\frac{|b_5|}{19} + \frac{|b_5|}{2^{19}\times 19}.$$

   Remark: Maple gives $L(f) = 1.13\ldots$, and an upper bound of the error committed when approximating $L(f)$ by the first five terms is 0.00144.

Theorem 1 (*Dominated Convergence Theorem*). *Let I be a real interval with endpoints a and b in $\overline{\mathbb{R}}$ (with $a < b$). Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of piecewise continuous functions on I such that:*

- *the sequence of functions $(f_n)_{n\in\mathbb{N}}$ converges pointwise on I to a piecewise continuous function f,*
- *there exists a piecewise continuous function g on I such that*

$$\forall t \in I,\ \forall n \in \mathbb{N},\ \big|f_n(t)\big| \le g(t),$$

  and, in the case where I is not a closed and bounded interval, the improper integral

$$\int_a^b g(t)\,\mathrm{d}t$$

  converges (the domination condition*).*

Then

$$\lim_{n\to+\infty} \int_a^b \big|f_n(t) - f(t)\big|\,\mathrm{d}t = 0,$$

and in particular,

$$\lim_{n\to+\infty} \int_a^b f_n(t)\,\mathrm{d}t = \int_a^b f(t)\,\mathrm{d}t,$$

and, in the case where I is not a closed and bounded interval, the improper integral on the right-hand side is absolutely convergent.

Exercise 2 (*11.5 points*)**.** In this exercise, we work in $\mathbb{R}^2$ with its standard dot product and with the standard Euclidean norm. The first two parts are independent from each other.

Part I – *4.5 points*

1. Let q be the quadratic form defined by:

$$q : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x,y) & \longmapsto & 3x^2 + 2xy + 3y^2. \end{array}$$

   a) Determine an orthonormal basis $\mathscr{B}'$ in which the matrix of q is diagonal.

b) We define

$$\mathscr{S} = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$

Let $u \in \mathscr{S}$. We denote by

$$[u]_{\mathscr{B}'} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

the coordinates of the vector u in the basis $\mathscr{B}'$. Show that $x'^2 + y'^2 = 1$ and deduce that $2 \leq q(u) \leq 4$.

c) Show that there exists $u_m \in \mathscr{S}$ such that $q(u_m) = 2$ and that there exists $u_M \in \mathscr{S}$ such that $q(u_M) = 4$. Deduce the value of $\max\limits_{u \in \mathscr{S}} q(u)$ and the value of $\min\limits_{u \in \mathscr{S}} q(u)$.

2. Let f be the mapping defined by:

$$f : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x,y) & \longmapsto & \frac{2}{3}(x+y)^3 + 2(x+y)^2 + (x-y)^2. \end{array}$$

a) Determine the critical points of f.

b) Show that f admits a unique local minimum, and that this local minimum is attained at a unique point $N \in \mathbb{R}^2$ that you will determine. Is this local minimum a global minimum of f on $\mathbb{R}^2$?

c) Let q_N be the quadratic form, the matrix of which, in the standard basis of $\mathbb{R}^2$ is the Hessian matrix $H_N f$ of f at the point N. Determine the value of $\max\limits_{u \in \mathscr{S}} q_N(u)$ and the value of $\min\limits_{u \in \mathscr{S}} q_N(u)$.

Part II – *3.5 points*

Let $a, b \in \mathbb{R}$ and $R \in \mathbb{R}_+^*$. Let f be the mapping defined by:

$$f : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x,y) & \longmapsto & (x-a)^2 + (y-b)^2. \end{array}$$

We define the circle $\mathscr{C}$ centered at (a,b) of radius R:

$$\mathscr{C} = \{(x,y) \in \mathbb{R}^2 \mid f(x,y) = R^2\}.$$

1. For what points $(x,y) \in \mathscr{C}$ do we have $\partial_2 f(x,y) = 0$? what do we observe for the tangent line to the circle $\mathscr{C}$ at these points?

2. In this question and the next one, $(x_0, y_0) \in \mathscr{C}$ is such that $\partial_2 f(x_0, y_0) \neq 0$.

a) Justify that there exists an open interval $I \subset \mathbb{R}$ such that $x_0 \in I$, an open interval $J \subset \mathbb{R}$ such that $y_0 \in J$ and a function $\psi : I \longrightarrow J$ of class C^2 such that

$$\forall (x,y) \in I \times J, \ \Big((x,y) \in \mathscr{C} \iff y = \psi(x)\Big).$$

b) What is the value of $\psi(x_0)$?

c) Show that

$$\psi'(x_0) = -\frac{x_0 - a}{y_0 - b}.$$

d) Show that

$$\psi''(x_0) = -\frac{1 + \psi'(x_0)^2}{y_0 - b}.$$

e) Deduce that

$$R = \frac{\big(1 + \psi'(x_0)^2\big)^{3/2}}{|\psi''(x_0)|}.$$

3. Determine an equation of the tangent line to the circle $\mathscr{C}$ at the point (x_0, y_0).

The notion of curvature enables us to quantify how much the graph of a function of one variable recedes from its tangent line at a point. Let I be an real interval and let $h : I \longrightarrow \mathbb{R}$ be a function of class C^2. We admit that we can define the curvature of h at a point $t \in I$ by the number

$$\frac{h''(t)}{\big(1 + h'(t)^2\big)^{3/2}}.$$

The goal of the next part is to define a similar notion of curvature for the graph of a function of 2 variables.

Part III – *3.5 points*
Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function of class C^2. Let (x_0, y_0) be a critical point of f. Let $v = (v_1, v_2) \in \mathbb{R}^2$. We define the mapping

$$g_v : \begin{array}{rcl} \mathbb{R} & \longrightarrow & \mathbb{R} \\ t & \longmapsto & f(x_0 + tv_1, y_0 + tv_2). \end{array}$$

1. Compute $g'_v(t)$ and $g''_v(t)$ in terms of the partial derivatives of f.

2. Let ρ be the mapping defined by

$$\rho : \begin{array}{rcl} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ v & \longmapsto & \dfrac{g''_v(0)}{\left(1 + g'_v(0)^2\right)^{3/2}}. \end{array}$$

   Deduce from the previous questions a simple expression of $\rho(v)$. The number $\rho(v)$ is called *the curvature of the function f in the direction (x_0, y_0) at the critical point (x_0, y_0)*.

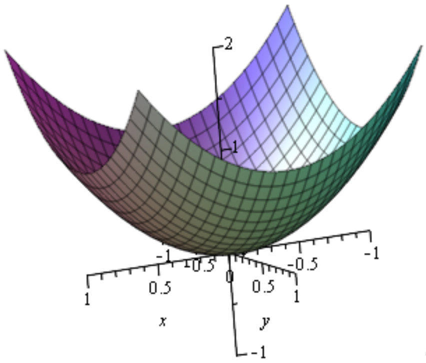
3. Check that the mapping ρ is in fact the quadratic form the matrix of which in the standard basis of $\mathbb{R}^2$ is the Hessian matrix of f at the point (x_0, y_0).

4. Determine the maximal curvature $\max\limits_{\|v\|=1} \rho(v)$ as well as the minimal curvature $\min\limits_{\|v\|=1} \rho(v)$ *of the function f defined in Part I at the point N* as well as the directions in which they are attained.

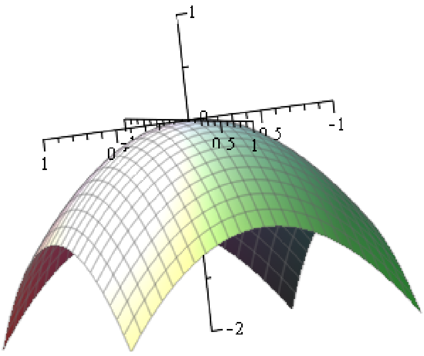
5. Back to the general case. We define the Gaussian curvature of f at the critical point (x_0, y_0) as the product of the eigenvalues of its Hessian matrix at the point (x_0, y_0).

   a) Justify that the Gaussian curvature is a real number.

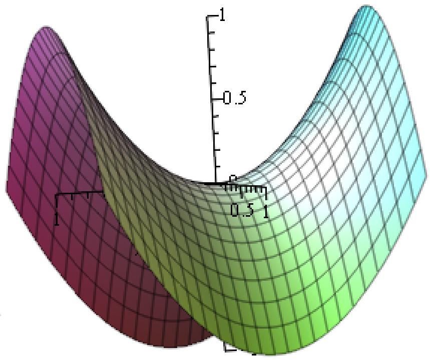
   b) Determine the sign of the Gaussian curvature of the functions f, given by the graphs below, at the critical point $(0, 0)$, which is *non-degenerate* (that is, the Hessian matrix is invertible at the point $(0, 0)$).



(i)



(ii)



(iii)

   c) What is the value of the Gaussian curvature at the critical point $(0, 0)$ of the function f given by the graph below?