SCAN 2 — Solution of Math Test #1

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Exercise 1.

1. The function ln is continuous on $(0,1]$, hence the integral I is improper at 0. Let $X \in (0,1]$. Then:

$$I_X = \int_X^1 \ln(t)\,\mathrm{d}t = \big[t\ln(t) - t\big]_{t=X}^{t=1} = -1 - X\ln(X) + X.$$

   Now, $\lim_{X\to 0^+} X\ln(X) = 0$, hence $\lim_{X\to 0^+} I_X = -1 \in \mathbb{R}$, hence the improper integral I is convergent and $I = -1$.

2. The function $t \mapsto \mathrm{e}^{-t}\ln(t)$ is continuous on $(0,+\infty)$. Hence the integral J is improper at 0 and at $+\infty$.

   - *Convergence at* 0*:* since $\mathrm{e}^{-t}\ln(t) \underset{t\to 0^*}{\sim} \ln(t) < 0$ and since, by Question 1, the improper integral

$$\int_0^1 \ln(t)\,\mathrm{d}t$$

     is convergent, we conclude that J is convergent at 0.
   - *Convergence at* $+\infty$*:* we know that $\lim_{t\to+\infty} t^2\mathrm{e}^{-t}\ln(t) = 0$, hence there exists $A > 1$ such that

$$\forall t > A,\ 0 < t^2\mathrm{e}^{-t}\ln(t) < 1,$$

     hence

$$\forall t > A,\ 0 < \mathrm{e}^{-t}\ln(t) < \frac{1}{t^2}.$$

     Since (by Riemann at $+\infty$ with $\alpha = 2 > 1$) the improper integral

$$\int_1^\infty \frac{\mathrm{d}t}{t^2}$$

     converges at $+\infty$, we conclude, by the comparison test, that J converges at $+\infty$.

   Hence J is a convergent integral.

3. The function $t \mapsto \mathrm{e}^{-t}\ln(t)\cos(t)$ is continuous on $(0,+\infty)$, hence the integral K is improper at $+\infty$.

   - *Convergence at* 0*:* it is similar to the convergence at 0 of J: since $\mathrm{e}^{-t}\ln(t)\cos(t) \underset{t\to 0^*}{\sim} \ln(t) < 0$, by the equivalent test, K converges at 0.
   - *Convergence at* $+\infty$*:* observe that

$$\forall t > 1,\ 0 \leq \left|\mathrm{e}^{-t}\ln(t)\cos(t)\right| \leq \mathrm{e}^{-t}\ln(t)$$

     and since, by Question 2 the improper integral

$$\int_1^{+\infty} \mathrm{e}^{-t}\ln(t)\,\mathrm{d}t$$

     is convergent at $+\infty$ we conclude, by the Comparison Test, that K is absolutely convergent at $+\infty$, hence K is convergent at $+\infty$.

   Hence K is a convergent integral.

Exercise 2.

1. Let $p \in \mathbb{R}$ and $\omega \in \mathbb{R}_+^*$. The functions $t \mapsto \cos(\omega t)\mathrm{e}^{-pt}$ and $t \mapsto \sin(\omega t)\mathrm{e}^{-pt}$ are continuous on $[0,+\infty)$, hence the integrals I and J are improper at $+\infty$.

   Let $X > 0$ and define

$$I_X = \int_0^X \cos(\omega t)\mathrm{e}^{-pt}\,\mathrm{d}t \qquad \text{and} \qquad J_X = \int_0^X \sin(\omega t)\mathrm{e}^{-pt}\,\mathrm{d}t.$$

Then,

$$\begin{aligned}
I_X + iJ_X &= \int_0^X \big(\cos(\omega t) + i\sin(\omega t)\big)\mathrm{e}^{-pt}\,\mathrm{d}t \\
&= \int_0^X \mathrm{e}^{i\omega t}\mathrm{e}^{-pt}\,\mathrm{d}t \qquad \textit{by Euler's Formula} \\
&= \int_0^X \mathrm{e}^{(-p+i\omega)t}\,\mathrm{d}t \\
&= \left[\frac{\mathrm{e}^{(-p+i\omega)t}}{-p+i\omega}\right]_{t=0}^{t=X} \qquad \textit{since } -p+i\omega \neq 0 \\
&= \frac{\mathrm{e}^{(-p+i\omega)X} - 1}{-p+i\omega}.
\end{aligned}$$

Now observe that since $p > 0$, one has $\left|\mathrm{e}^{(-p+i\omega)X}\right| = \mathrm{e}^{-pX} \underset{X\to+\infty}{\longrightarrow} 0$ hence

$$\lim_{X\to+\infty} (I_X + iJ_X) = -\frac{1}{-p+i\omega} = \frac{p+i\omega}{p^2+\omega^2}.$$

Since I_X and J_X are real, we conclude that

$$\lim_{X\to+\infty} I_X = \frac{p}{p^2+\omega^2} \qquad \text{and} \qquad \lim_{X\to+\infty} J_X = \frac{\omega}{p^2+\omega^2},$$

hence I and J converge and

$$I = \frac{p}{p^2+\omega^2} \qquad \text{and} \qquad J = \frac{\omega}{p^2+\omega^2},$$

2. Let $\alpha \in \mathbb{R}$. The function $t \mapsto \dfrac{\mathrm{e}^{-t}-1}{t^\alpha}$ is continuous on $(0,+\infty)$ hence the integral I_α is improper at 0 and at $+\infty$.

   - *Convergence at* 0*:* we have the following equivalent:

     $$\frac{\mathrm{e}^{-t}-1}{t^\alpha} \underset{t\to0^+}{\sim} \frac{-t}{t^\alpha} = -\frac{1}{t^{\alpha-1}} < 0,$$

     and we know, by Riemann, that the improper integral

     $$\int_0^1 \frac{\mathrm{d}t}{t^{\alpha-1}}$$

     converges (at 0) if and only if $\alpha - 1 < 1$ i.e., $\alpha < 2$. Hence, by the equivalent test, the improper integral I_α converges at 0 if and only if $\alpha < 2$.
   - *Convergence at* $+\infty$*:* since $\mathrm{e}^{-t} - 1 \underset{t\to+\infty}{\longrightarrow} -1$, we have the following equivalent:

     $$\frac{\mathrm{e}^{-t}-1}{t^\alpha} \underset{\to+\infty}{\sim} -\frac{1}{t^\alpha} < 0,$$

     and we know, by Riemann, that the improper integral

     $$\int_1^{+\infty} \frac{\mathrm{d}t}{t^\alpha}$$

     converges (at $+\infty$) if and only if $\alpha > 1$. Hence, by the equivalent test, the improper integral I_α converges at $+\infty$ if and only if $\alpha > 1$.

   Conclusion: the improper integral I_α converges if and only if $1 < \alpha < 2$.

Exercise 3.

1. Let φ be the linear mapping

$$\begin{array}{rccc}
\varphi : & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\
& (x,y) & \longmapsto & (2x-y, x-y).
\end{array}$$

Then

$$[\varphi]_{\text{std}} = \begin{pmatrix} 2 & -1 \\ 1 & -1 \end{pmatrix},$$

and $\det \varphi = -1 \neq 0$ hence φ is invertible. It is clear that

$$\forall u \in \mathbb{R}^2,\ N(u) = \big\|\varphi(u)\big\|_1,$$

hence N is a norm on $\mathbb{R}^2$.

2. Moreover, for $u \in \mathbb{R}^2$,

$$\begin{aligned} u \in B &\iff N(u) \leq 1 \\ &\iff \big\|\varphi(u)\big\|_1 \leq 1 \\ &\iff \varphi(u) \in B_{\|\cdot\|_1} \\ &\iff u \in \varphi^{-1}(B_{\|\cdot\|_1}) \end{aligned}$$

hence $B = \varphi^{-1}(B_{\|\cdot\|_1})$. Now,

$$[\varphi^{-1}]_{\text{std}} = \begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix},$$

and we can understand graphically how B is obtained from $B_{\|\cdot\|_1}$: the ball $B_{\|\cdot\|_1}$ consists of the square with vertices $(0,1)$, $(1,0)$, $(-1,0)$ and $(0,-1)$, hence the ball B consists of the parallelogram obtained by applying φ^{-1} to this square. To obtain it explicitly, we only need to determine the image of the two vertices $(0,1)$ and $(1,0)$ by φ, and use the fact that B is symmetric with respect to $(0,0)$:

$$\varphi^{-1}\big((1,0)\big) = (1,1), \qquad \varphi^{-1}\big((0,1)\big) = (-1,-2).$$

We hence obtain the ball B shown on Figure 3.

Exercise 4.

1. Let $f_0 \in E$.

a) Let $h \in E$. Then

$$\big|\varphi(h)\big| = \left|\int_0^1 f_0(t)h(t)\,\mathrm{d}t\right| \leq \int_0^1 \big|f_0(t)\big|\big|h(t)\big|\,\mathrm{d}t \leq \int_0^1 \big|f_0(t)\big|\|h\|_\infty\,\mathrm{d}t = \|h\|_\infty \int_0^1 \big|f_0(t)\big|\,\mathrm{d}t = \|h\|_\infty\|f_0\|_1.$$

Moreover,

$$\|f_0\|_1 = \int_0^1 \big|f_0(t)\big|\,\mathrm{d}t \leq \int_0^1 \|f_0\|_\infty\,\mathrm{d}t = \|f_0\|_\infty,$$

hence $\|f_0\|_1\|h\|_\infty \leq \|f_0\|_\infty\|h\|_\infty$.

b) We notice that φ is a linear mapping, hence φ is continuous if and only if φ is continuous at 0_E. Now, from the previous inequality and the Squeeze Theorem, we conclude that:

$$\lim_{\|h\|_\infty \to 0} \big|\varphi(h)\big| = 0.$$

2. Let $f_0 \in E$ and let's show that Ψ is differentiable at f_0 and determine $\mathrm{d}_{f_0}\Psi$: let $h \in E$. Then:

$$\Psi(f_0+h) = \int_0^1 \big(f_0(t)+h(t)\big)^2\,\mathrm{d}t = \int_0^1 \big(f_0(t)^2+2f_0(t)h(t)+h(t)^2\big)\,\mathrm{d}t = \int_0^1 f_0(t)^2\,\mathrm{d}t+2\int_0^1 f_0(t)h(t)\,\mathrm{d}t+\int_0^1 h(t)^2\,\mathrm{d}t = \Psi(f_0)$$

where φ is the linear mapping of the previous question. Now we know (we already showed it in the previous question) that: $\|h^2\|_1 \leq \|h^2\|_\infty$, and $\|h^2\|_\infty = \|h\|_\infty^2$. Hence

$$\frac{\|h^2\|_1}{\|h\|_\infty} \leq \|h\|_\infty \underset{\|h\|_\infty \to 0}{\longrightarrow} 0.$$

Since φ is linear and continuous, we conclude that Ψ is differentiable at f_0 and that

$$\begin{aligned} \mathrm{d}_{f_0}\Psi \ :\ E &\longrightarrow \mathbb{R} \\ h &\longmapsto 2\int_0^1 f_0(t)h(t)\,\mathrm{d}t. \end{aligned}$$

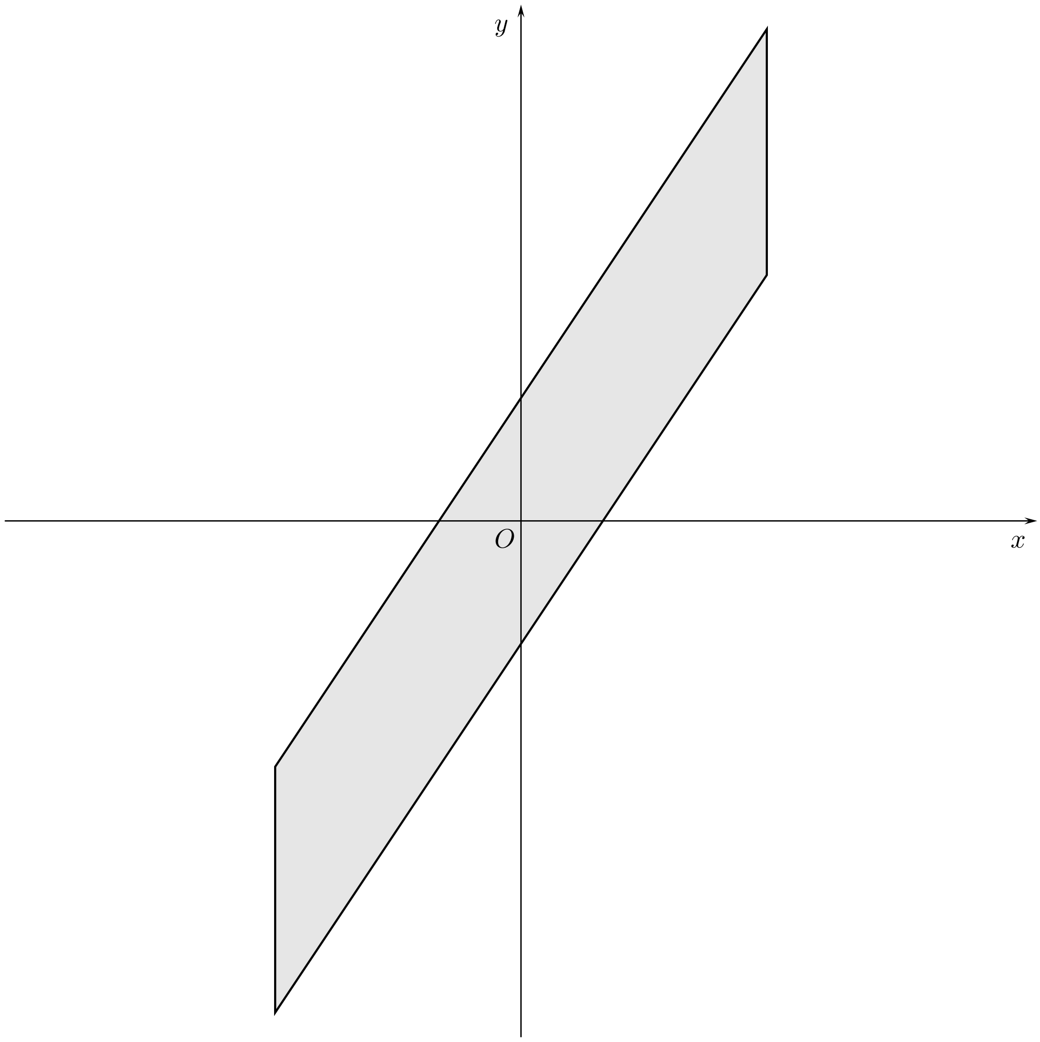

Figure 1. Ball B of Exercise 3

Exercise 5. Let $(x, y) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Then

$$\left|f(x,y)\right| = \frac{|xy|^{3/2}}{x^2+2y^2} \leq \frac{|xy|^{3/2}}{x^2+y^2} \leq \frac{\left(\|(x,y)\|_2\|(x,y)\|_2\right)^{3/2}}{\|(x,y)\|_2^2} = \|(x,y)\|_2 \underset{\|(x,y)\|_2\to 0}{\longrightarrow} 0.$$

Hence $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$.

Exercise 6.

1. - Let $u \in E$ such that $N(u) = 0$. Then we must have, e.g., $N_2(u) = 0$ and hence, since N_2 is a norm on E we must have $u = 0_E$.
   - Let $u, v \in E$. Then since N_1 and N_2 are norms on E we have

     $$N_1(u+v) \leq N_1(u) + N_1(v) \qquad \text{and} \qquad N_2(u+v) \leq N_2(u) + N_2(v)$$

     Hence, by the Property $(*)$ we also have $N(u+v) \leq \max\{N_1(u)+N_1(v), N_2(u)+N_2(v)\} \leq \max\{N_1(u), N_2(u)\} + \max\{N_1(v), N_2(v)\}$.
   - Let $u \in E$ and $\lambda \in \mathbb{R}$. Then since N_1 and N_2 are norms on E we have

     $$N_1(\lambda u) = |\lambda| N_1(u) \qquad \text{and} \qquad N_2(\lambda u) = |\lambda| N_2(u).$$

     Hence $\max\{N_1(\lambda u), N_2(\lambda u)\} = \max\{|\lambda| N_1(u), |\lambda| N_2(u)\} = |\lambda| \max\{N_1(u), N_2(u)\} = |\lambda| N(u)$.

2. Let $u \in E$. Then

$$\begin{aligned} u \in B &\iff N(u) \leq 1 \\ &\iff \max\{N_1(u), N_2(u)\} \leq 1 \\ &\iff N_1(u) \leq 1 \text{ and } N_2(u) \leq 1 \\ &\iff u \in B_1 \text{ and } u \in B_2 \\ &\iff u \in B_1 \cap B_2. \end{aligned}$$

Hence $B = B_1 \cap B_2$.

3. a) The ball B is the intersection of two squares, as shown on Figure 2.

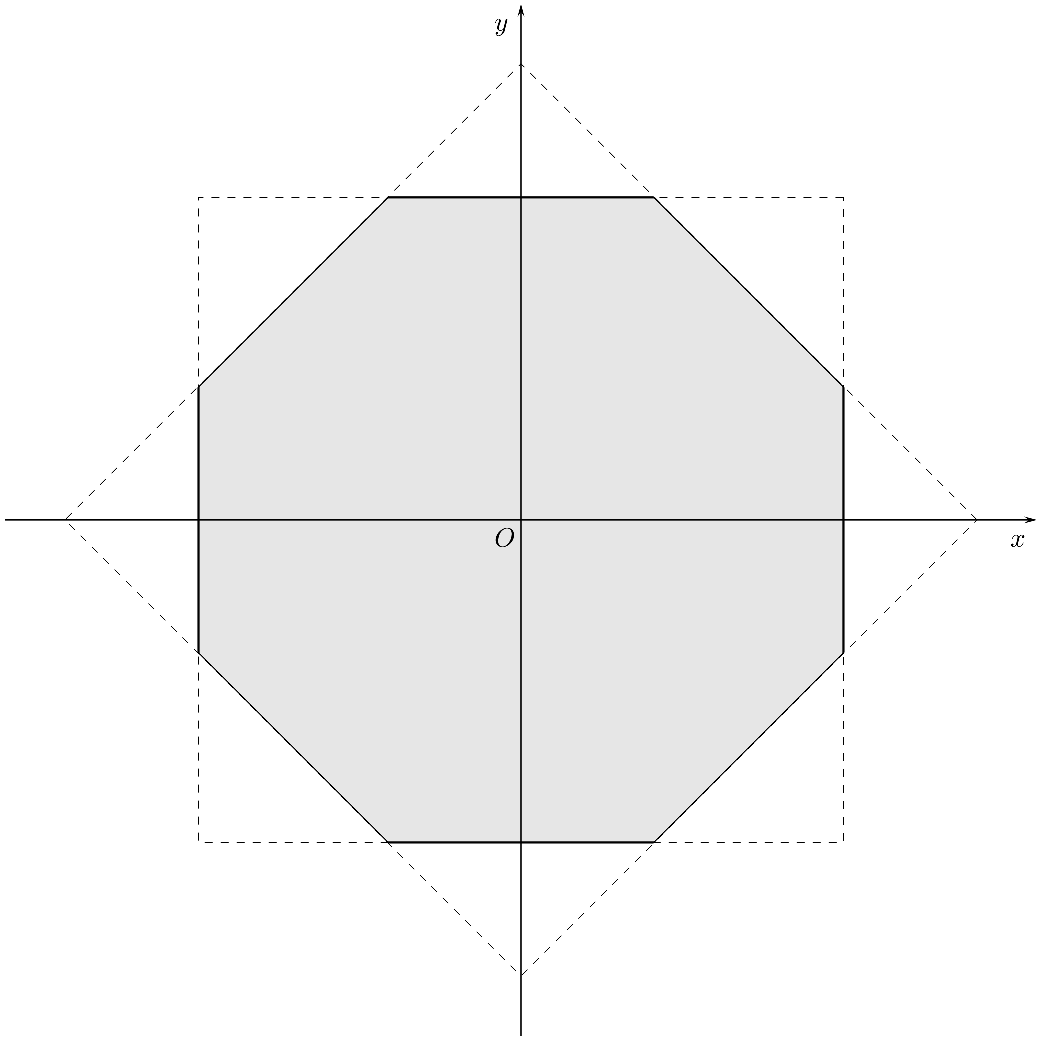

Figure 2. Ball B of Question 3a of Exercise 6

b) i) The ball B is the intersection of the ellipse and the parallelogram, as shown on Figure 3(b)i.

ii) The set B_P is the union of the ellipse and the parallelogram, as shown on Figure 3(b)ii. We notice on the figure that this set is not convex, hence P cannot be a norm on $\mathbb{R}^2$.
Hence, in general, P is not a norm on E.

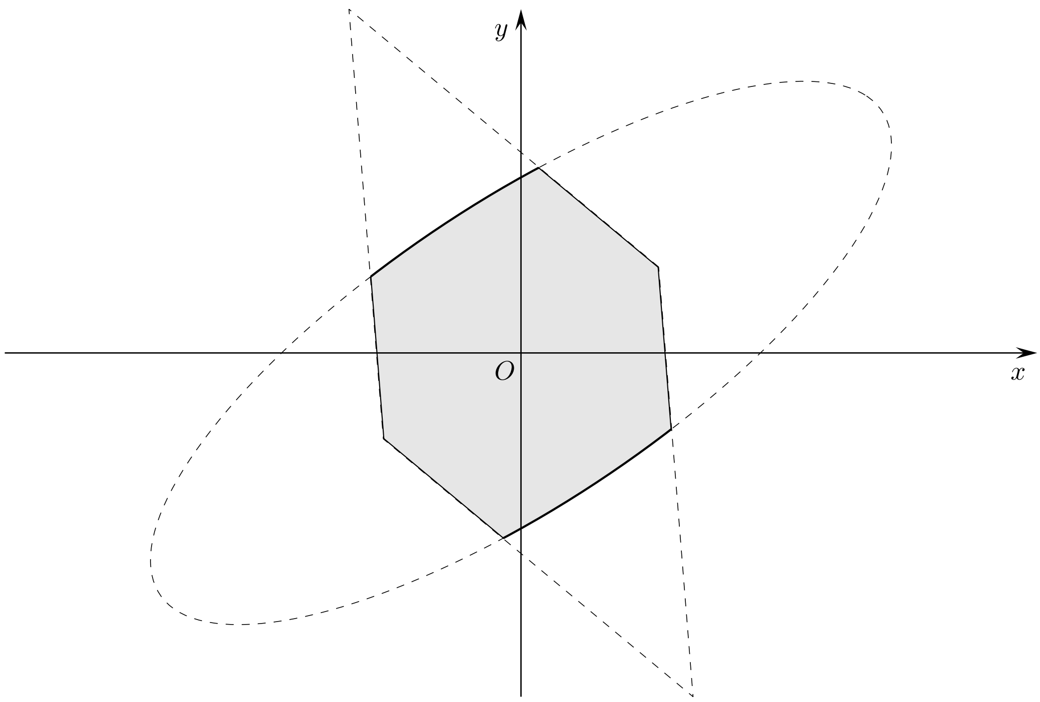

Figure 3. The ball B of Question 3(b)i of Exercise 6

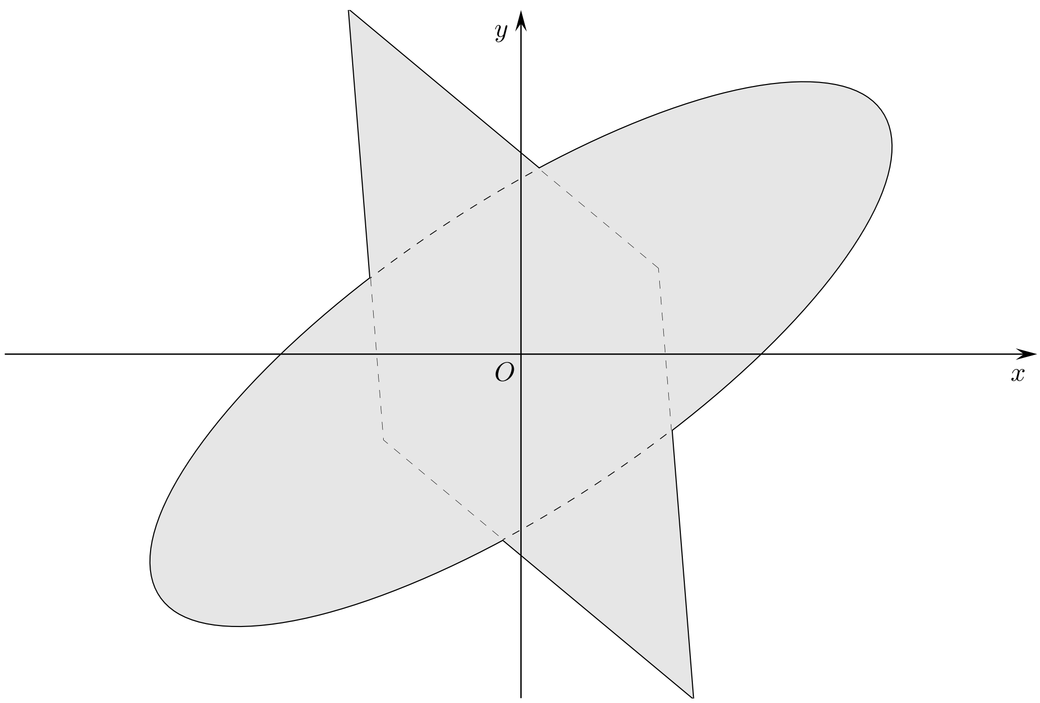

Figure 4. The set B_P of Question 3(b)ii of Exercise 6