INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — S3— Math Test #1 — 2h

November 3, 2016

No documents, no calculators, no cell phones or electronic devices allowed but you may keep your pet blobfish for moral support.

All your answers must be fully justified, unless noted otherwise.

Exercise 1. Don't forget to explicitly specify the points where the improper integrals are improper!

1. Show that the following improper integral converges and compute its value:

$$I = \int_0^1 \ln(t)\,\mathrm{d}t.$$

2. Show that the following improper integral converges:

$$J = \int_0^{+\infty} \mathrm{e}^{-t}\ln(t)\,\mathrm{d}t.$$

3. Deduce that the following improper integral converges:

$$K = \int_0^{+\infty} \mathrm{e}^{-t}\ln(t)\cos(t)\,\mathrm{d}t.$$

Exercise 2. The two questions of this exercise are independent from each other:

1. Let $p, \omega \in \mathbb{R}_+^*$. Show that the following improper integrals are convergent and determine their values:

$$I = \int_0^{+\infty} \cos(\omega t)\mathrm{e}^{-pt}\,\mathrm{d}t \qquad \text{and} \qquad J = \int_0^{+\infty} \sin(\omega t)\mathrm{e}^{-pt}\,\mathrm{d}t.$$

Hint: you may want to use complex numbers and compute $I + iJ$.

2. Let $\alpha \in \mathbb{R}$. Determine the values of α for which the following improper integral is convergent:

$$I_\alpha = \int_0^{+\infty} \frac{\mathrm{e}^{-t}-1}{t^\alpha}\,\mathrm{d}t.$$

Exercise 3. Let

$$N : \begin{array}{rcl} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x,y) & \longmapsto & |2x-y| + |x-y|. \end{array}$$

1. Show that N is a norm on $\mathbb{R}^2$.
2. Plot the closed unit ball B of N.

Exercise 4. Let $E = C\big([0,1],\mathbb{R}\big)$ be the vector space of continuous functions on $[0,1]$. We equip E with the norm $\|\cdot\|_\infty$.

1. Let $f_0 \in E$ and define the mapping

$$\varphi : \begin{array}{rcl} E & \longrightarrow & \mathbb{R} \\ h & \longmapsto & \displaystyle\int_0^1 f_0(t)h(t)\,\mathrm{d}t. \end{array}$$

   a) Show that

$$\forall h \in E,\ \big|\varphi(h)\big| \le \|f_0\|_1\|h\|_\infty \le \|f_0\|_\infty\|h\|_\infty.$$

   b) Deduce that φ is continuous.

2. We define the mapping

$$\Psi : \begin{array}{rcl} E & \longrightarrow & \mathbb{R} \\ f & \longmapsto & \displaystyle\int_0^1 f(t)^2\,\mathrm{d}t. \end{array}$$

Show that Ψ is differentiable on E, and determine the differential $\mathrm{d}_{f_0}\Psi$ of Ψ at a point $f_0 \in E$.

Exercise 5. We define the function f as

$$\begin{array}{rccc} f : & \mathbb{R}^2 \setminus \{(0,0)\} & \longrightarrow & \mathbb{R} \\ & (x,y) & \longmapsto & \dfrac{|xy|^{3/2}}{x^2+2y^2}. \end{array}$$

Show that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists and determine its value.

Exercise 6. *In a question of this exercise, the following result (that you don't need to justify) may be useful:*

$$(*) \qquad \forall a_1,a_2,b_1,b_2,c_1,c_2 \in \mathbb{R},\ \Big(a_1 \le b_1 + c_1 \text{ and } a_2 \le b_2 + c_2 \implies \max\{a_1,a_2\} \le \max\{b_1,b_2\} + \max\{c_1,c_2\}\Big).$$

Let E be a real vector space and let N_1 and N_2 be two norms on E. We denote by B_1 and B_2 the closed unit balls of N_1 and N_2 respectively.
We define the mappings N and P on E as:

$$\begin{array}{rccc} N : & E & \longrightarrow & \mathbb{R}_+ \\ & u & \longmapsto & \max\big\{N_1(u), N_2(u)\big\} \end{array} \quad \text{and} \quad \begin{array}{rccc} P : & E & \longrightarrow & \mathbb{R}_+ \\ & u & \longmapsto & \min\big\{N_1(u), N_2(u)\big\}. \end{array}$$

1. Show that N is a norm on E.
2. Explain how the closed unit ball B of N is obtained from B_1 and B_2.
3. Explain how the set
$$B_P = \{u \in E \mid P(u) \le 1\}$$
is obtained from B_1 and B_2.
4. a) Plot the closed unit ball B of N in the case where $N_1 = \dfrac{1}{\sqrt{2}}\|\cdot\|_1$ and $N_2 = \|\cdot\|_\infty$.

   b) On Figure 1 we have represented two (closed) sets B_1 and B_2 in $\mathbb{R}^2$, that correspond to two unit balls of norms N_1 and N_2 respectively.

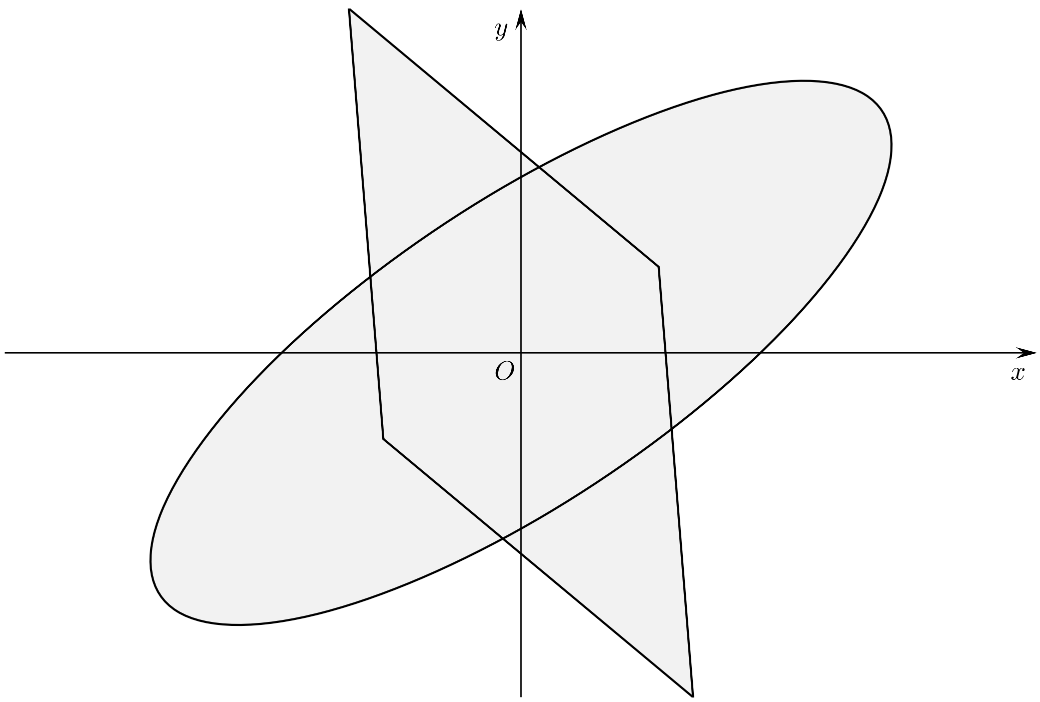

Figure 1 – Balls B_1 (the parallelogram) and B_2 (the ellipse) of Question 4b of Exercise 6

   i) Plot the closed unit ball B of N.

   ii) Plot the set B_P. Conclude that, in general, P is not a norm on E.